INSTANTANEOUS COMPACTIFICATION OF THE SUPPORT OF SOLUTIONS FOR NONLINEAR DIFFUSION-REACTION EQUATIONS

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This report is devoted to the study of some qualitative properties of solutions for a wide class of nonlinear partial differential equations. We consider the Cauchy problem for quasilinear parabolic second order equation with degenerating absorption coefficient − weight function (potential $h(t)$), the presence of which play the important role in the study qualitative properties of solutions.

Investigation object – instantaneous compactification property of support of the solutions of parabolic equation.

The goal of the work – to prove that Cauchy problem for quasilinear parabolic second order equation of divergent type has instantaneous compactification property of support of the solutions and to get the estimates of radius of compactification.

So, we study the behavior of the energy (generalized) solutions of following Cauchy problem for the next type of a quasilinear parabolic equation with the model representative:

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-1} \frac{\partial u}{\partial x_i} \right) + h(t)|u|^{\lambda-1} u = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $\nabla u = grad\, u = \frac{\partial u}{\partial x_1 \ldots \partial x_n}$; p and λ are positive real numbers. Assume that $u_0(x) \in \mathrm{L}_q(\mathbb{R}^n)$, $q > 1$ and $h(t)$ is a continuous, nonnegative function, such that $h(0) = 0$.

Definition. The function $u(x,t) \in C(\mathrm{L}_q(\mathbb{R}^n); (0,\mathrm{T})) \cap \mathrm{L}_{q+\lambda-1}(\mathbb{R}^n \times (0,\mathrm{T}))$ is energy (generalized) solution (1), (2), if

$$|u|^{\frac{q-2}{p+1}} u_{x_i} \in L_{p+1}\left(R^n \times (0,T)\right), \quad u_{x_i} \in L_{\frac{p+1}{p}}\left(R^n \times (0,T)\right), \tag{3}$$

and satisfies integral identity:

$$\int_{R^n} |u|^q \phi(x,T)\,dx - \int_{R^n} |u_0| \phi(x,0)\,dx = q \int_{R^n \times (0,T)} \sum_{i=1}^{n} |\nabla u|^{p-1} \frac{\partial u}{\partial x_i} \left(|u|^{q-2} u\phi \right)_{x_i} dxdt +$$

$$+\int\limits_{R^n\times(0,T)} |u|^q \phi_t dxdt - q \int\limits_{R^n\times(0,T)} h(t)|u|^{q+\lambda-1}\phi(x,t)\,dxdt \qquad (4)$$

for $\forall\,\phi(x,t)\in C^1(\mathbb{R}^n\times(0,T))$, such that ϕ and ϕ_t are bounded and, besides, $\nabla\phi$ has compact support.

Note here, that the existence of solutions in the above sense is well known from papers Bernis (1986 and 1988) in the case

$$\frac{\partial}{\partial t}\left(|u|^{q-1}u\right)+Au+b(x,t,u)=0,$$

where $(x,t)\in\mathbb{R}^n\times(0,T_0),\ T_0\le\infty,\ n\ge 1,\ q>0;\ b(x,t,u)=|u|^{\lambda-1}u$ and operator A has variation structure.

So, as mentioned above, the main focus of the work is investigation of some inner properties of solutions of a large class of nonlinear diffusion–reaction equations.

Definition. The problem (1), (2) has the instantaneous compactification property of support of the solutions (briefly, the ICSS property) if for any $t>0$ the support of an energy (generalized) solution $u(x,t)$ is bounded even if it is unbounded for $t=0$.

The work of Evans and Knerr (1979) was the first where the ICSS property was systematically investigated in the case

$$u_t-\Delta u+b(u)=0,\quad b(0)=0,\ b(\mathrm{s})>0\quad \forall s>0.$$

The authors found the precise conditions on the behavior of the function $b(u)$ in the neighborhood of 0, which provide ICSS property for any nonnegative, continuous bounded initial functions $u_0(x)$: $u_0(x)\to 0,\quad |x|\to\infty$. From the papers Kersner, Nicolosi (1992) and Kalashnikov (1993) we have results about ICSS property for one-dimensional equations like

$$u_t-\left(u^m\right)_{xx}+g(x)u^p=0,\ m\ge 1,\ \ p\in(0,1),$$

which was established under the following conditions:

$$u_0\le c_0\left(1+|x|\right)^{-\gamma},\quad g(x)\ge c_1\left(1+|x|\right)^{-\beta},\quad 0<\beta<\gamma(1-p),\quad \gamma>0,\quad \mathrm{c}_i>0.$$

It should be noted that this phenomenon (ICSS property) may occur in other important physical models. For example, Gilding and Kersner (1990) for the equation

$$u_t=\left(u^m\right)_{xx}+\left(u^n\right)_x,\ 0<n<1,\ m\ge 1$$

had received the following theorem: if $u_0(x)\sim cx^{-\frac{1}{1-n}}$ when $x\to\infty$, then $u(x,t)>0,\ t\in\left(0,\ \frac{1}{n}c^{1-n}\right),\quad x\ge x_0>0$ and $u(x,t)$ has a compact support for $t>\frac{1}{n}c^{1-n}$. Hence the phenomenon of instantaneous compactification holds under

$u_0 = o\left(x^{-\frac{1}{1-n}} \right)$. Analogous results were establish for the first order hyperbolic equation $u_t - \left(u^n\right)_x = 0,\ 0 < n < 1$. Besides, for variational inequalities the ICSS pro-perty was investigated. But all of the mentioned results have been obtained for non-negative solution of second order equations and with assumption on initial function $u_0(x)$: $u_0(x) \to 0,\ |x| \to \infty$ or $u_0(x)$ has a majorant. The main technique was the comparison principle (or maximum principle). To sum up, note here, that if $u_0(x)$ does not have monotone majorant, then even for the simplest equation such as $u_t - u_{xx} + u^p = 0,\ 0 < p < 1$ we cannot give answer about the behavior of solution as the maximum principle is inadequate.

Method has been utilized throughout work is called "energy method" (see works Diaz, Veron (1985); Antontsev, Diaz, Shmarev (1995); Veron, Shishkov (2007)). The idea of this approach consists getting special inequalities linking different energy norms of solution. The analysis of these inequalities leads to the necessary results. Universality is one of the advantages of the energy method, because it can be applied to the higher order equations. This approach finds its reflection in next theorem concerning the ICSS property of Cauchy problem for the mentioned above equation such as (1).

Theorem. Let $u(x,t)$ – is energy (generalized) solution of the problem (1), (2). Then in both of the cases:

(a) $p \geq 1,\ 0 < \lambda < 1$

(b) $0 < \lambda < p$, where $\dfrac{n-2}{n+2} < p < 1$ for $n > 2$; and $0 < p < 1$ for $n \leq 2$

the problem (1), (2) has the ICSS property (e.g., solution $u(x,t)$ of the problem (1), (2) is bounded for $t > 0$.)

We would like to remark, that the proof of this theorem is based on the method in the spirit of paper Kersner and Shishkov (1996):

Step 1. In the integral identity (4) we substitute $\phi(x,t) = u(x,t)\eta^{p+1}(x,t)$ (where $\eta \geq 0$ is special cut-off function), integrating by parts and after some transformations we make replacement: $w = |u|^{\frac{q-2}{p+1}} u$. Next, we apply Young's, Holder, Gagliardo-Nirenberg interpolation inequalities.

Step 2. Prove, that $\forall \tau > 0 \quad \exists s(\tau) < \infty$:

$$H_T(\tau,s) := E_T(\tau,s) + I_T(\tau,s) = 0 \ \Rightarrow w = 0 \Rightarrow u(x,t) = 0, \qquad (5)$$

$E_T(\tau,s) := \int\limits_{\Omega(s)\times(\tau,T)} |w|^{\alpha}\, dxdt \neq 0$ and $I_T(\tau,s) := \int\limits_{\Omega(s)\times(\tau,T)} |w|^{p+1}\, dxdt \neq 0.$

In order to complete proof of (5) enough to check three points:

a) $H_T(\tau,s)$ is non-negative function as $w := |u|^{\frac{q-2}{p+1}} u$, $|w|^{\alpha} \geq 0$, $|w|^{p+1} \geq 0$, hence, $E_T(\tau,s) \geq 0$ and $I_T(\tau,s) \geq 0$. To sum up, we have that $H_T(\tau,s) \geq 0$.

b) $H_T(\tau,s)$ is non-increasing function as $G_\tau^T(s) \to 0$, when $s \to \infty$, $|w|^{\alpha} \geq 0$, $|w|^{p+1} \geq 0$, hence, $E_T(\tau,s)$ and $I_T(\tau,s)$ are non-increasing functions.

c) finally, we have to check inequality from analogue of lemma Stampakiya

$$H\left(\tau + H^{\alpha}(\tau,s),\ s + H^{\beta}(\tau,s)\right) \leq \delta H(\tau,s), \quad 0 < \delta < 1.$$

The analysis of these inequalities leads to the necessary results.

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