MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

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Guidelines
to practical tasks on the indefinite integral on the academic discipline
"MATHEMATICAL ANALYSIS AND LINEAR ALGEBRA"

for full-time foreign students and students taught in English of training direction 6.030601 "Management" specialization "Business Administration"

Kharkiv
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2016

Затверджено на засіданні кафедри вищої математики й економіко-математичних методів.

Протокол № 8 від 04.03.2016 р.

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G 94 **Guidelines** to practical tasks on the indefinite integral on the academic discipline "Mathematical Analysis and Linear Algebra" for full-time foreign students and students taught in English of training direction 6.030601 "Management" specialization "Business Administration" / compiled by Ie. Misiura. – Kharkiv : S. Kuznets KhNUE, 2016. – 48 p. (English)

The sufficient theoretical material on the academic discipline and typical examples are presented to help students master the material on the theme "The indefinite integral" and apply the obtained knowledge to practice. Individual tasks for self-study work and a list of theoretical questions are given to promote the improvement and extension of students' knowledge on the theme.

Recommended for full-time foreign students and students taught in English of training direction 6.030601 "Management" specialization "Business Administration".

Introduction

Integral calculus plays a very important role in economics in particular in problems concerning the optimum management and plans. Therefore the deep knowledge of this section of mathematical analysis and linear algebra is necessary for modern economists.

In the guidelines, only the most principal topics of integral calculus are stated in brief.

The present guidelines are the continuation of the part where the notions of differential calculus were discussed. By means of these notions, the notions of the indefinite integral and the antiderivative of the function being the most fundamental ones in mathematics can be introduced.

Guidelines for the Indefinite Integral

Let us consider the inverse problem: the derivative of a function is given, it is necessary to find this function. In our case we are given a function $f(x)$, which is the derivative of a certain function $F(x)$, that is $F'(x)=f(x)$ an it is required to restore the function $F(x)$.

Definition. The function is *an antiderivative* (or a primitive function) of the function $f(x)$ on an open interval (a,b) at any point of this interval. The function $F(x)$ is differentiable, where $F'(x)=f(x)$. Instead of the open interval (a,b) we can consider a closed interval or a half-interval.

Let us as see whether or not the antlderivative is determined for a function in a unique manner.

Let us formulate the key theorem on antiderivatives.

Theorem. Every continuous function possesses an indefinite number of antiderivatives, any two of them only differing by a constant.

Proof. Let $F_1(x)$ and $F_2(x)$ be antiderivatives of a function $f(x)$ on an open interval (a,b). Then $F_1'(x)=f(x)$ and $F_2'(x)=f(x)$ on this interval, that is

$$[F_2(x)-F_1(x)]' = f(x)-f(x)=0$$

and the function $F(x)=F_2(x)-F_1(x)$ is constant: $F(x)=C$, that is, $F_2(x)=F_1(x)+C$. On the other hand, if C is an arbitrary constant, and $F(x)$ is an antiderivative of the function $f(x)$ of the interval (a,b), then obviously, the function $F(x)+C$ is also an antiderivative of the function $f(x)$ on the interval (a,b). Thus the entire set of antiderivatives of the function $f(x)$ on the interval (a,b) is contained among the functions of the form $F(x)+C$, where C is an arbitrary constant.

Definition. The collection of all antiderivatives of the function $f(x)$ is called *the indefinite integral* of $f(x)$ (written as: $\int f(x)dx$). The sign $\int$ is termed *the integral sign*, the expression $f(x)dx$ is *the integrand expression*, the function $f(x)$ is *the integrand* (or the integrand function), and the variable x is *the integration variable*.

We will write $\int f(x)dx = F(x)+C$ if any corresponding antiderivatives differ by a constant: $F(x)=F(x)+C$.

Let us note the following obvious properties of indefinite integrals:

1) a derivative of an indefinite integral equals an integrand:

$$\left(\int f(x)dx\right)' = f(x);$$

2) a differential of an indefinite integral equals an integrand expression:

$$d\left(\int f(x)dx\right) = f(x)dx;$$

3) if $f(x)dx = dF(x)$, then $\int dF(x) = F(x)+C$;

4) the indefinite integral of a sum (or difference) of a finite number of functions is a sum (or difference) of the indefinite integrals of these functions:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

and applying the rule for differentiating a sum to the right member of equality we obtain:

$$\left(\int [f(x)\pm g(x)]dx\right)' = \left(\int f(x)dx\right)' \pm \left(\int g(x)dx\right)' = f(x)\pm g(x),$$

which coincides with the integrand on the left-hand side of the equality, i.e. with its derivative;

5) a constant factor in the integrand can be taken outside the sign of the indefinite integral:

$$\int k\cdot f(x)dx = k\cdot\int f(x)dx \text{, where } k = const\,.$$

Indeed, the derivative of the right-hand side of the equality, i.e.

$$\left(k\cdot\int f(x)dx\right)' = kf(x) \quad \text{and} \quad \left(\int kf(x)dx\right)' = kf(x)$$

is equal to the derivative of the left-hand side;

6) $\int f(kx)dx = \frac{1}{k}F(kx)+C$.

Indeed,

$$\left(\int f(kx)dx\right)' = f(kx),\ \left(\frac{1}{k}F(kx)+C\right)' = \frac{1}{k}F'(kx)+0 = \frac{1}{k}\cdot k\cdot f(kx) = f(kx),$$

that is the derivative on the left-hand side is equal to the derivative on the right-hand side;

7) $\int f(kx+b)dx = \frac{1}{k}F(kx+b)+C$, where $kx+b$ is a linear function.

Indeed,

$$\left(\int f(kx+b)dx\right)' = f(kx+b),$$

$$\left(\frac{1}{k}F(kx+b)+C\right)' = \frac{1}{k}F'(kx+b)+0 = \frac{1}{k}\cdot k\cdot f(kx+b) = f(kx+b).$$

The Theorem of the Invariance of the Integration Formula

Let $\int f(x)dx = F(x) + C$ be any given integration formula and $u = \varphi(x)$ be any function possessing a continuous derivative. Then

$$\int f(u)dx = F(u) + C .$$

Proof. From $\int f(x)dx = F(x) + C$ it follows that $F'(x) = f(x)$. Now take the function $F(u) = F(\varphi(x))$ by the theorem on the invariance of the form of the first differential of a function, its differential is

$$dF(u) = F'(u)du = f(u)du .$$

Hence, $\int f(u)du = \int dF(u) = F(u)$.

The Basic Table of Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$.
2. $\int \frac{dx}{x} = \ln|x| + C$.
3. $\int dx = x + C$.
4. $\int 0 \cdot dx = C$.
5. $\int a^x dx = \frac{a^x}{\ln a} + C$, $(a > 0, a \neq 1)$.
6. $\int e^x dx = e^x + C$.
7. $\int \sin x dx = -\cos x + C$.
8. $\int \cos x dx = \sin x + C$.
9. $\int \mathrm{tg} x dx = -\ln|\cos x| + C$.
10. $\int \mathrm{ctg} x dx = \ln|\sin x| + C$.

11. $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$.

12. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$.

13. $\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$.

14. $\int \frac{dx}{\cos x} = \ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$.

15. $\int \frac{dx}{1+x^2} = \begin{cases} \operatorname{arctg} x + C \\ -\operatorname{arcctg} x + C \end{cases}$.

16. $\int \frac{dx}{a^2+x^2} = \begin{cases} \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \\ -\frac{1}{a}\operatorname{arcctg}\frac{x}{a} + C \end{cases}, \ (a>0)$.

17. $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C \\ -\arccos x + C \end{cases}$

18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \begin{cases} \arcsin\frac{x}{a} + C \\ -\arccos\frac{x}{a} + C \end{cases}, \ (a>0)$.

19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C, \ (a \neq 0)$.

20. $\int \frac{dx}{\sqrt{x^2 \pm a}} = \ln\left|x + \sqrt{x^2 \pm a}\right| + C, \ (a \neq 0)$.

Integration with the aid of the table of basic indefinite integrals is called *the direct integration*.

Let's consider two examples of this method.

Example 1. Find the integral:

$$\int\left(2x^2 + \frac{7}{2x^3} - e^x\right)dx = \left| \text{apply the property} : \frac{1}{x^3} = x^{-3} \right| =$$

$$= \int \left(2x^2 + \frac{7}{2} \cdot x^{-3} - e^x \right) dx = \left| \text{use tabular integrals 1 and 6} \right| =$$

$$= 2\frac{x^{2+1}}{2+1} + \frac{7}{2} \cdot \frac{x^{-3+1}}{-3+1} - e^x + C = \frac{2}{3}x^3 - \frac{7}{4}x^{-2} - e^x + C.$$

Example 2. Find the integral: $\int \cos^2 3x dx$.

Solution. Here we reduce the degree of the integrand, i.e. $\cos^2 3x = \frac{1}{2}(1 + \cos 6x)$ and substitute it under the sign of integral

$$\int \cos^2 3x dx = \int \frac{1}{2}(1 + \cos 6x) dx = \frac{1}{2}\int (1 + \cos 6x) dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 6x dx =$$

applying tabular integrals 3 and 6 and property 6 and we get

$$= \frac{1}{2}x + \frac{1}{2} \cdot \frac{1}{6}\sin 6x + C = \frac{1}{2}x + \frac{1}{12}\sin 6x + C.$$

Change of the Variable (Substitution) in the Indefinite Integral

Let the function $F(x)$ be an antiderivative of the function $f(x)$ on a closed interval $[a, b]$, and let the function $x = \varphi(t)$ attains values on the interval $[a, b]$ for $t \in [\alpha, \beta]$. If the derivative of $\varphi(t)$ exists, then, by the formula for differentiation of a composite function, we have:

$$F'(\varphi(t)) = F'(x) \quad \text{or} \quad F'(\varphi(t)) = f(\varphi(t))\varphi'(t).$$

Therefore, the function $F(\varphi(t))$ is an antiderivative of the function $f(\varphi(t))\varphi'(t)$ for $t \in [\alpha, \beta]$ that is

$$\int f(\varphi(t))\varphi'(t) dt = \int F'(\varphi(t)) dt = F(\varphi(t)) + C \quad \text{for} \quad t \in [\alpha, \beta].$$

Let us put $F(\varphi(t))+C=\int f(x)dx$ if $x=\varphi(t)$ and therefore, we get the following formula for change of variables in the indefinite integral:

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt=\int f(\varphi(t))d\varphi(t).$$

Example 3. Find the integral: $\int \frac{2xdx}{x^2+1}$.

Solution. If we suggest that $x^2+1=t$ then the numerator $2xdx$ is readily dt and the integral is reduced to

$$\int \frac{2xdx}{x^2+1}=\left|\begin{array}{l} t=x^2+1 \\ dt=t'dx \\ dt=\left(x^2+1\right)'dx \\ dt=2xdx \end{array}\right|=\int \frac{dt}{t}=$$

It is the tabular integral, then

$$=\ln|t|+C=$$

Let us get back to the previous variable by substitution $x^2+1=t$:

$$=\ln\left|x^2+1\right|+C.$$

Example 4. Find the integral: $\int \sin^3 x\cos xdx$.

Solution. Because $d(\sin x)=\cos xdx$ and assuming $t=\sin x$ we transform the integrand expression $\sin^3 x\cos xdx=\sin^3 xd(\sin x)=t^3dt$ then

$$\int \sin^3 x\cos xdx=\left|\begin{array}{l} t=\sin x,\ dt=t'dx \\ dt=(\sin x)'dx,\ dt=\cos xdx \end{array}\right|=\int t^3dt=$$

It is the tabular integral, then

$$=\frac{t^4}{4}+C=$$

Let us get back to the previous variable by substitution $t=\sin x$:

$$=\frac{1}{4}\sin^4 x+C.$$

Example 5. Find the integral: $\int\sqrt{1-x^2}dx$.

Solution. Let $x=\sin t$, then $dx=(\sin t)'dt=\cos tdt$, substitute it in the integrand expression

$$\int\sqrt{1-x^2}dx=\int\sqrt{1-(\sin t)^2}\cos tdt=\int\sqrt{(\cos t)^2}\cos tdt=\int\cos^2 tdt=$$

applying the formula $\cos^2 t=\frac{1+\cos 2t}{2}$ we get

$$=\int\frac{1+\cos 2t}{2}dt=\frac{1}{2}\int(1+\cos 2t)dt=\frac{1}{2}\int dt+\frac{1}{2}\int\cos 2tdt=\frac{1}{2}t+\frac{1}{2}\cdot\frac{1}{2}\sin 2t+C=$$

returning to x we must use that $t=\arcsin x$:

$$=\frac{1}{2}\arcsin x+\frac{1}{4}\sin(2\arcsin x)+C=$$

and we know that $\sin 2\alpha=2\sin\alpha\cos\alpha$, then

$$\sin(2\arcsin x)=2\sin(\arcsin x)\cos(\arcsin x)$$

and $\sin(\arcsin x)=x$, $\cos(\arcsin x)=\sqrt{1-\sin^2(\arcsin x)}=\sqrt{1-x^2}$, we write

$$=\frac{1}{2}\arcsin x+\frac{1}{2}\sin(\arcsin x)\cos(\arcsin x)+C=\frac{1}{2}\arcsin x+\frac{1}{2}x\sqrt{1-x^2}+C.$$

In many cases the substitution $t=\psi(x)$, where $\psi(x)$ has the inverse function $x=\varphi(t)$, is more convenient.

Example 6. Find the integral: $\int\frac{dx}{\sin x}$.

Solution. Representing $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$ and using the change of variables $\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\mathrm{tg}\frac{x}{2}=t$, where the denominator of the integrand is divided and multiplied simultaneously by $\cos\frac{x}{2}$, we get

$$\int\frac{dx}{\sin x}=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{\cos\frac{x}{2}dx}{2\sin\frac{x}{2}\cos^2\frac{x}{2}}=\int\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\frac{dx}{2\cos^2\frac{x}{2}}=$$

where $\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\frac{1}{t}$ and $dt=\left(tg\frac{x}{2}\right)'dt=\frac{1}{2}\frac{1}{\cos^2\frac{x}{2}}dx=\frac{dx}{2\cos^2\frac{x}{2}}$, we obtain

$$=\int\frac{1}{t}dt=$$

It is the tabular integral, then

$$=\ln|t|+C=$$

Let us get back to the previous variable by substitution $t=\mathrm{tg}\frac{x}{2}$:

$$=\ln\left|\operatorname{tg}\frac{x}{2}\right|+C.$$

Example 7. Find the integral: $\int\frac{dx}{\cos x}$.

Solution. Assuming $x+\frac{\pi}{2}=t$, $\left(x+\frac{\pi}{2}\right)'dx=dt$ or $dx=dt$ and writing $\cos x=\sin\left(x+\frac{\pi}{2}\right)$ we obtain the integral that is analogous to the previous

$$\int\frac{dx}{\cos x}=\int\frac{dx}{\sin\left(x+\frac{\pi}{2}\right)}=\int\frac{dt}{\sin t}=\ln\left|\operatorname{tg}\frac{t}{2}\right|+C=\ln\left|\operatorname{tg}\frac{1}{2}\left(x+\frac{\pi}{2}\right)\right|+C=$$

$$=\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|+C.$$

The integrals 4 and 5 should be remembered because they are frequently used as table integrals.

It is necessary to pay attention to the following: if the numerator of the integrand is the differential of the denominator then the substitution $t=f(x)$ just gives the result, i.e.

$$\int\frac{f'(x)dx}{f(x)}=\int\frac{df(x)}{f(x)}=\int\frac{dt}{t}=\ln|t|+C=\ln|f(x)|+C.$$

Integration by Parts

Let us prove the validity of the formula

$$\int u(x)v'(x)dx=u(x)v(x)-\int u'(x)v(x)dx$$

assuming that both sides of the equality have sense. For this purpose, it is sufficient to evaluate the indefinite integral of both sides of the equality

$$u(x)v'(x)=[u(x)v(x)]'-u'(x)v(x).$$

Let us give a simple notation of the formula for integration by parts

$$\int u(x)dv(x)=u(x)v(x)-\int v(x)du(x).$$

There are classes of integrals which are found by means of this method. This method is applied to three types of integrals.

Type	No.	Kind of integral	The factor u	The factor dv
The first type	1	$\int P_n(x)\cdot a^x dx$, where $P_n(x)$ is a polynomial	$P_n(x)$	$a^x dx$
	2	$\int P_n(x)\cdot e^x dx$	$P_n(x)$	$e^x dx$
	3	$\int P_n(x)\cdot \sin x dx$	$P_n(x)$	$\sin x dx$
	4	$\int P_n(x)\cdot \cos x dx$	$P_n(x)$	$\cos x dx$
The second type	5	$\int P_n(x)\cdot \arccos x dx$	$\arccos x$	$P_n(x)dx$
	6	$\int P_n(x)\cdot \arcsin x dx$	$\arcsin x$	$P_n(x)dx$
	7	$\int P_n(x)\cdot \mathrm{arctg} x dx$	$\mathrm{arctg} x$	$P_n(x)dx$
	8	$\int P_n(x)\cdot \mathrm{arcctg} x dx$	$\mathrm{arcctg} x$	$P_n(x)dx$
	9	$\int P_n(x)\cdot \ln x dx$	$\ln x$	$P_n(x)dx$
The third type	10	$\int a^x \cdot \sin x dx$	$\sin x$ or a^x	$a^x dx$ or $\sin x dx$
	11	$\int e^x \cdot \sin x dx$	$\sin x$ or e^x	$e^x dx$ or $\sin x dx$
	12	$\int a^x \cdot \cos x dx$	$\cos x$ or a^x	$a^x dx$ or $\cos x dx$
	13	$\int e^x \cdot \cos x dx$	$\cos x$ or e^x	$e^x dx$ or $\cos x dx$

For instance, let us consider $\int P_n(x)\cdot \arccos x dx$, $\int P_n(x)\cdot \arcsin x dx$, $\int P_n(x)\cdot \mathrm{arctg} x dx$, $\int P_n(x)\cdot \mathrm{arcctg} x dx$, $\int P_n(x)\cdot \ln x dx$ and others.

Example 8. Find the integral: $\int \frac{\ln x}{\sqrt[3]{x}} dx$.

Solution. It is the integral of the second type:

$$u = \ln x,\ dv = \frac{dx}{\sqrt[3]{x}}.$$

Then

$$du = (\ln x)' dx = \frac{1}{x} dx, \quad v = \int dv = \int \frac{dx}{\sqrt[3]{x}} = \int x^{-\frac{1}{3}} dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{2} x^{\frac{2}{3}}.$$

Let's substitute functions into the formula $\int u \cdot dv = uv - \int v \cdot du$ and get:

$$\int \frac{\ln x}{\sqrt[3]{x}} dx = \frac{3}{2} x^{\frac{2}{3}} \ln x - \int \frac{3}{2} x^{\frac{2}{3}} \frac{dx}{x} = \frac{3}{2} x^{\frac{2}{3}} \ln x - \frac{3}{2} \int x^{-\frac{1}{3}} dx =$$

$$= \frac{3}{2} x^{\frac{2}{3}} \ln x - \frac{3}{2} \cdot \frac{3}{2} x^{\frac{2}{3}} + C = \frac{3}{2} x^{\frac{2}{3}} \ln x - \frac{9}{4} x^{\frac{2}{3}} + C.$$

Example 9. Find the integral: $\int x \cos 3x dx$.

Solution. It is the integral of the first type: $u = x,\ dv = \cos 3x dx$.

Let's find $du = dx,\ v = \int \cos 3x dx = \frac{1}{3} \sin 3x$ (suppose that $C = 0$).

Let's substitute functions into the formula $\int u \cdot dv = uv - \int v \cdot du$ and get:

$$\int x \cos 3x dx = x \cdot \frac{1}{3} \sin 3x - \int \frac{1}{3} \sin 3x dx.$$

Let's apply the table of the basic integrals and obtain:

$$\int x \cos 3x dx = \frac{x}{3} \sin 3x - \frac{1}{3}\left(-\frac{1}{3} \cos 3x\right) + C = \frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x.$$

There are cases where we must use the rule of integration by parts several times.

Example 10. Find the integral: $\int x^2 \sin x dx$.

Solution. It is the integral of the first type.

$$\int x^2 \sin x dx = \begin{cases} u = x^2 & dv = \sin x dx \\ du = 2x dx & v = -\cos x \end{cases} = -x^2 \cos x + \int \cos x \cdot 2x dx =$$

$$= \begin{cases} u = x & dv = \cos x dx \\ du = dx & v = \sin x \end{cases} = -x^2 \cos x + 2\left(x \sin x - \int \sin x dx\right) =$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

Example 11. Find the integral: $\int e^x \sin x dx$.

Solution. It is the integral of the third type.

Let's assume $u = e^x, \quad dv = \sin x dx$, then $du = e^x dx, \quad v = -\cos x dx$ and

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx.$$

We apply the rule of integration by parts to $\int e^x \cos x dx$ once more, suggesting $u = e^x, \quad dv = \cos x dx$ and $du = e^x dx, \quad v = \sin x$ we rewrite as:

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx.$$

Here we have obtained $-\int e^x \sin x dx$. Transferring this integral into the left-hand side, we get

$$2\int e^x \sin x dx = -e^x \cos x + e^x \sin x + C$$

or $\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$.

Example 12. Find the integral: $\int \operatorname{arctg} x dx$.

Solution. Let us denote $u = \operatorname{acrtg} x, dv = dx$, then we obtain $du = \frac{dx}{1+x^2}, v = x$, then

$$\int \operatorname{arctg} x dx = x \operatorname{arctg} x - \int \frac{x}{1+x^2} dx = x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx =$$

Here we see that $2xdx = d\left(x^2+1\right)$, then we write:

$$\int \operatorname{arctg} x dx = x \operatorname{arctg} x - \int \frac{x}{1+x^2} dx = x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx =$$

$$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{d\left(1+x^2\right)}{1+x^2} dx = x \operatorname{arctg} x - \frac{1}{2} \ln\left|1+x^2\right| + C.$$

Integration of Rational Functions with a Quadratic Trinomial

The rational functions with a quadratic trinomial are the functions of the kinds:

$$\int \frac{A}{ax^2+bx+c} dx, \int \frac{Ax+B}{ax^2+bx+c} dx, \int \frac{A}{\sqrt{ax^2+bx+c}} dx, \int \frac{Ax+B}{\sqrt{ax^2+bx+c}} dx,$$

where ax^2+bx+c is the quadratic trinomial.

Let's consider the first and second integrals: $\int \frac{A}{ax^2+bx+c} dx$, $\int \frac{A}{\sqrt{ax^2+bx+c}} dx$. They are reduced to tabular integrals if you allocate the perfect square in the denominator with the help of the formulas:

$$(y \pm z)^2 = y^2 \pm 2yz + z^2$$

(the square of the sum or the square of the difference).

Example 13. Find the integral: $\int \frac{2}{\sqrt{x^2+8x+25}}dx$.

Solution. Let's allocate the perfect square in the denominator with the help of the formula $(y+z)^2 = y^2 + 2yz + z^2$ and get:

$$x^2+8x+25 = x^2+2\cdot 4\cdot x+16+9 = \underbrace{x^2+2\cdot 4\cdot x+4^2}+9 = (x+4)^2+9.$$

Let's substitute:

$$\int \frac{2}{\sqrt{x^2+8x+25}}dx = \int \frac{2}{\sqrt{(x+4)^2+9}}dx = 2\cdot\int \frac{dx}{\sqrt{(x+4)^2+9}}.$$

Let's use the tabular integral 20. We have

$$2\cdot\int \frac{dx}{\sqrt{(x+4)^2+9}} = 2\cdot\ln\left|\left(x+4+\sqrt{(x+4)^2+9}\right)\right|+C =$$

$$= 2\cdot\ln\left|\left(x+4+\sqrt{(x+4)^2+9}\right)\right|+C = 2\cdot\ln\left|\left(x+4+\sqrt{x^2+8x+25}\right)\right|+C.$$

Let's consider the third and fourth integrals: $\int \frac{Ax+B}{ax^2+bx+c}dx$, $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$. To integrate these functions we should use the following rules:

1) to allocate the perfect square in the trinomial with the help of the formulas $(y \pm z)^2 = y^2 \pm 2yz + z^2$ (the square of the sum or the square of the difference) and obtain a new denominator $(x \pm p)^2 \pm q$;

2) to apply the substitution:

$$t = x + p \qquad\qquad t = x - p$$
$$dt = dx \qquad\qquad dt = dx$$
$$x = t - p \qquad\qquad x = t + p;$$

3) to present the initial integral as a sum of two integrals, the first one is the tabular integral and the second one may be integrated by substitution.

Let's consider four kinds of such integrals:

a) $\int \frac{xdx}{\sqrt{a - x^2}} = -\sqrt{a - x^2} + C$, $(a \neq 0)$;

b) $\int \frac{xdx}{a - x^2} = -\frac{1}{2}\ln\left|x^2 - a\right| + C$, $(a \neq 0)$;

c) $\int \frac{xdx}{\sqrt{a - x^2}} = -\sqrt{a - x^2} + C$, $(a \neq 0)$;

d) $\int \frac{xdx}{\sqrt{x^2 \pm a}} = \sqrt{x^2 \pm a} + C$, $(a \neq 0)$;

4) to get back to the previous variable by substitution:

$$t = x + p, \qquad\qquad t = x - p.$$

Let's give examples.

Example 14. Let's find this integral:

$$\int \frac{(7 - 8x)dx}{x^2 - 6x + 2} = \int \frac{(7 - 8x)dx}{(x - 3)^2 - 7} = \left| \begin{array}{l} t = x - 3 \\ dt = dx \\ x = t + 3 \end{array} \right| = \int \frac{(7 - 8(t + 3))dt}{t^2 - 7} = \int \frac{(10 - 8t)dt}{t^2 - 7} =$$

$$= 10\int \frac{dt}{t^2 - 7} - 8\int \frac{tdt}{t^2 - 7} = 10\int \frac{dt}{t^2 - \left(\sqrt{7}\right)^2} - 8\int \frac{tdt}{t^2 - 7} = \frac{10}{2\sqrt{7}}\ln\left|\frac{t - \sqrt{7}}{t + \sqrt{7}}\right| -$$

$$-\frac{8}{2}\ln\left|t^2 - 7\right| + C = \frac{5}{\sqrt{7}}\ln\left|\frac{x - 3 - \sqrt{7}}{x - 3 + \sqrt{7}}\right| - \frac{8}{2}\ln\left|x^2 - 6x + 2\right| + C.$$

Example 15. Find the indefinite integral: $\int \frac{2xdx}{3x^2+5x+4}$.

Solution. We take out the common factor 3 and allocate the perfect square in the denominator:

$$3x^2+5x+4=3\left(x^2+\frac{5}{3}x+\frac{4}{3}\right)=3\left(x^2+2\cdot\frac{5}{6}x+\frac{25}{36}-\frac{25}{36}+\frac{4}{3}\right)=$$

$$=3\left(x^2+2\cdot\frac{5}{6}x+\frac{25}{36}+\frac{23}{36}\right)=3\left(\left(x+\frac{5}{6}\right)^2+\frac{23}{36}\right).$$

Let's make the substitution: $t=x+\frac{5}{6}$, then $x=t-\frac{5}{6}$, $dx=dt$.

We have

$$\int\frac{2xdx}{3x^2+5x+4}=\int\frac{2xdx}{3\left(\left(x+\frac{5}{6}\right)^2+\frac{23}{36}\right)}=\frac{1}{3}\int\frac{2\left(t-\frac{5}{6}\right)}{t^2+\frac{23}{36}}dt=\frac{1}{3}\int\frac{2t-\frac{5}{3}}{t^2+\frac{23}{36}}dt=$$

$$=\frac{1}{3}\int\frac{2tdt}{t^2+\frac{23}{36}}-\frac{5}{9}\int\frac{dt}{t^2+\frac{23}{36}}=\frac{1}{3}\ln\left(t^2+\frac{23}{36}\right)-\frac{5}{9}\cdot\frac{6}{\sqrt{23}}\operatorname{arctg}\frac{6t}{\sqrt{23}}+C_1=$$

We get back to the previous variable. Thus,

$$=\frac{1}{3}\ln\left(\left(x+\frac{5}{6}\right)^2+\frac{23}{36}\right)-\frac{5}{9}\cdot\frac{6}{\sqrt{23}}\operatorname{arctg}\frac{6\left(x+\frac{5}{6}\right)}{\sqrt{23}}+C_1=$$

$$=\frac{1}{3}\ln\left(x^2+\frac{5}{3}x+\frac{4}{3}\right)-\frac{5}{9}\cdot\frac{6}{\sqrt{23}}\operatorname{arctg}\frac{6x+5}{\sqrt{23}}+C_1=$$

$$=\frac{1}{3}\ln\left(3x^2+5x+4\right)-\frac{10}{3\sqrt{23}}\operatorname{arctg}\frac{6x+5}{\sqrt{23}}+C.$$

Integration of the Rational Function

The simplest types of indefinite integrals representable in terms of elementary functions are integrals of the rational function, that is, integrals of functions of the form $R(x)=\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

If the degree of the polynomial $P(x)$ is greater than, or equal to the degree of the polynomial $Q(x)$ (rational function is an improper fraction), we first divide by the polynomial $Q(x)$, using the rules for dividing polynomials

$$\frac{P(x)}{Q(x)}=P_1(x)+\frac{P_2(x)}{Q(x)}.$$

Here $P_1(x)$ and $P_2(x)$ are polynomials, the degree of the polynomial $P_2(x)$ being less than the degree of the polynomial $Q(x)$.

The integral $\int P_1(x)dx$ is readily computed. Thus, when finding the integral $\int \frac{P_2(x)}{Q(x)}dx$ we may always see if the degree of the polynomial $P_2(x)$ is less than the degree of the polynomial $Q(x)$.

Further we simplify the expression $\frac{P_2(x)}{Q(x)}$, assuming that the polynomials $P_2(x)$ and $Q(x)$ have real coefficients of the powers of the variable x.

It is possible to simplify the expression for a rational function if the polynomial $Q(x)$ can be written as a product of linear and quadratic factors with real coefficients

$$Q(x)=a_0(x-x_1)^{k_1}\cdot(x-x_2)^{k_2}\cdot\ldots\cdot(x-x_r)^{k_r}\times$$
$$\times\left(x^2+p_1x+q_1\right)^{l_1}\ldots\cdot\left(x^2+p_sx+q_s\right)^{l_S}, \qquad (1)$$

where $x_1, x_2, \ldots, x_r$ being the roots and these roots are linear terms.

In this case the polynomial $Q(x)$ is said to have the roots $x = x_1$, $x = x_2, \ldots, x = x_r$ of multiplicity m. And the quadratic equations

$$x^2 + p_1 x + q_1 = 0, \ldots, x^2 + p_s x + q_s = 0,$$

have conjugate complex roots (that $p_i^2 - 4q_i < 0$) which are $2s$-fold conjugate complex roots of the same equations.

It will be assumed that factorization (1) has been determined, it turns out that the fraction $\dfrac{P_2(x)}{Q(x)}$ can be represented as a decomposition into partial fractions of the form

$$\frac{P_2(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{(x - x_1)^2} + \ldots + \frac{A_{k_1}}{(x - x_1)^{k_1}} + \ldots + \frac{B_1}{x - x_r} + \frac{B_2}{(x - x_r)^2} + \ldots + \frac{B_{k_r}}{(x - x_r)^{k_r}} +$$
$$+ \frac{C_1 x + D_1}{x^2 + p_1 x + q_1} + \frac{C_2 x + D_2}{\left(x^2 + p_1 x + q_1\right)^2} + \ldots + \frac{C_{l_1} x + D_{l_1}}{\left(x^2 + p_1 x + q_1\right)^{l_1}} + \ldots + \quad (2)$$
$$+ \frac{M_1 x + N_1}{x^2 + p_s x + q_s} + \frac{M_2 x + N_2}{\left(x^2 + p_s x + q_{s1}\right)^2} + \ldots + \frac{M_{l_s} x + N_{l_s}}{\left(x^2 + p_s x + q_s\right)^{l_s}},$$

where $A_1, A_2, \ldots, A_{k_1}, \ldots, N_1, \ldots, N_{l_s}$ are constants.

It follows that the integral of every rational fraction can be reduced to integrals of partial rational fractions of the indicated typed, namely:

1) $\displaystyle\int \frac{A}{x - a} dx$; 3) $\displaystyle\int \frac{Cx + D}{x^2 + px + q} dx$;

2) $\displaystyle\int \frac{A}{(x - a)^k} dx$; 4) $\displaystyle\int \frac{Cx + D}{\left(x^2 + px + q\right)^s} dx$, $s > 0$.

Thus when applying the method of integration of rational fractions presented here we proceed from the factorization of the denominator of the given fraction $\frac{P_2(x)}{Q(x)}$, to decomposition (2) and replace the given integral by the sum of the integrals of the corresponding partial fractions.

The finding of forth types of the integrals:

1) $\int \frac{A}{x-a}dx = A\ln|x-a| + C$;

2) $\int \frac{A}{(x-a)^k}dx = \int A(x-a)^{-k}dx = A\frac{(x-a)^{-k+1}}{-k+1} + C$;

3) in this case $\int \frac{Cx+D}{x^2+px+q}dx$ we apply the rule of integration of functions with a quadrate trinomial in the denominator. Here the first integral is tabular, the second is equal to one of four integrals, namely:

a) $\int \frac{xdx}{\sqrt{a-x^2}} = -\sqrt{a-x^2} + C$, $(a \neq 0)$;

b) $\int \frac{xdx}{a-x^2} = -\frac{1}{2}\ln\left|x^2-a\right| + C$, $(a \neq 0)$;

c) $\int \frac{xdx}{\sqrt{a-x^2}} = -\sqrt{a-x^2} + C$, $(a \neq 0)$;

d) $\int \frac{xdx}{\sqrt{x^2 \pm a}} = \sqrt{x^2 \pm a} + C$, $(a \neq 0)$;

4) at last we can find the integral using the method of the decrease of the power of the denominator and then the method of the integration by parts.

To determine the coefficients $A_1,\ A_2,\dots,\ A_{k_1},\dots,\ N_1,\dots,\ N_{l_s}$ we can use the following rule: the coefficients in line powers of x on both sides must be equal to each other. We can find all coefficients by this method. If the roots of the polynomial $Q(x)$ are real numbers there is a still simple method of determing these coefficients: putting in succession the roots of the polynomial $x = x_1,\ \ x = x_2,\dots, x = x_r$.

Thus the expression $\frac{P_2(x)}{Q(x)}$ can be represented in the form of the sum of the expressions of the following types: $\frac{A}{x-a}$ and $\frac{Cx+D}{x^2+px+q}$. The coefficients in the expansion of a rational function into common fractions are usually found by the method of undetermined coefficients: the supposed expansion is written with undetermined coefficients. Multiplying both sides of the equality by the common denominator, we get an equality between some polynomials. For undetermined coefficients a system of equations can be obtained either by equating the coefficients of the powers of the variable x on both sides of the equality or by using this equality for concrete values of the variable x.

Example16. Find the indefinite integral: $\int \frac{x^3+1}{x^3-x^2}dx$.

Here the integrand is the improper fraction because the power of the numerator is equal to the power of the denominator. Then for the first we divide the numerator by the denominator using the rule of the dividing of the polynomials and separate the integer part:

$$\begin{array}{r|l} x^3+1 & x^3-x^2 \\ \cline{2-2} -\ \underline{x^3-x^2} & 1 \\ x^2+1 & \end{array}.$$

Now the fraction is $\frac{x^3+1}{x^3-x^2}=1+\frac{x^2+1}{x^3-x^2}$. Thus,

$$\int \frac{x^3+1}{x^3-x^2}dx=\int\left(1+\frac{x^2+1}{x^3-x^2}\right)dx=\int dx+\int \frac{x^2+1}{x^3-x^2}dx.$$

Here we can represent the rest integral as the sum of partial fractions:

$$\frac{x^2+1}{x^2(x-1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1},$$

where A,B,C are undetermined coefficients.

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Reducing to the common denominator and equating the numerators on the both sides of this equality, we have

$$x^2+1=A(x-1)+Bx(x-1)+Cx^2.$$

Let us collect:

$$x^2+1=x^2(B+C)+x(A-B)+(-A).$$

and consider *the first method of finding of undetermined coefficients.*

To find the coefficients, we equate the coefficients at the equal powers of *x* on the left and on the right:

for x^2: $1=B+C$,

for x^1: $0=A-B$,

for x^0: $1=-A$.

We find the *undetermined coefficients:*
from the 3rd equation: $A=-1$;
from the 2nd equation: $A=B=-1$;
from the 1st equation: $C=1-B=1-(-1)=2$.

So,

$$\int\frac{x^3+1}{x^3-x^2}dx=\int dx+\int\frac{x^2+1}{x^3-x^2}dx=\int dx+\int\left(\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}\right)dx=$$

$$=\int dx+\int\left(\frac{-1}{x^2}+\frac{-1}{x}+\frac{2}{x-1}\right)dx=\int dx-\int\frac{1}{x^2}dx-\int\frac{1}{x}dx+2\int\frac{1}{x-1}dx=$$

$$=x+\frac{1}{x}-\ln|x|+2\ln|x-1|+C.$$

Example 17. Find the integral: $\int \frac{dx}{x^3-1}$.

Let us decompose the fraction into partial fractions:

$$\frac{1}{x^3-1}=\frac{1}{(x-1)\left(x^2+x+1\right)}=\frac{A}{x-1}+\frac{Bx+C}{\left(x^2+x+1\right)}.$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Then $$1=A\left(x^2+x+1\right)+(Bx+C)(x-1).$$

Let us collect:

$$1=x^2(A+B)+x(A-B+C)+(A-C).$$

Equating the factors at the identical degrees of x we obtain the coefficients:

for x^2: $0=A+B$,

for x^1: $0=A-B+C$,

for x^0: $1=A-C$.

We obtain the coefficients:

from the 1st equation: $B=-A$,

from the 2nd equation: $C=-A+B=-A-A=-2A$,

from the 3rd equation: $1=A-(-2A)$ or $1=3A$ or $A=\frac{1}{3}$,

then $B=-A=-\frac{1}{3}$, $C=-2A=-\frac{2}{3}$.

So, $$\int \frac{dx}{x^3-1}=\int\left(\frac{A}{x-1}+\frac{Bx+C}{\left(x^2+x+1\right)}\right)dx=\int\left(\frac{1/3}{x-1}+\frac{-1/3\,x-2/3}{\left(x^2+x+1\right)}\right)dx=$$

$$=\frac{1}{3}\int\frac{dx}{x-1}-\frac{1}{3}\int\frac{x+2}{x^2+x+1}dx=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx=\begin{vmatrix} t=x+\frac{1}{2} \\ dt=dx \\ x=t-\frac{1}{2}\end{vmatrix}=$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{t-\frac{1}{2}}{t^2+\frac{3}{4}}dt = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{t}{t^2+\frac{3}{4}}dt + \frac{1}{6}\int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\frac{1}{2}\ln\left|t^2+\frac{3}{4}\right| + \frac{1}{6}\frac{1}{\sqrt{3}/2}\operatorname{arctg}\frac{t}{\sqrt{3}/2} + C =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right| + \frac{1}{6}\frac{1}{\sqrt{3}/2}\operatorname{arctg}\frac{x+\frac{1}{2}}{\sqrt{3}/2} + C =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| + \frac{1}{3\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C.$$

Example 18. Find the integral: $\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}dx$.

The integrand is the proper fraction, let us decompose it into partial fractions.

$$\frac{15x^2-4x-81}{(x-3)(x+4)(x-1)} = \frac{A}{x-3} + \frac{B}{x+4} + \frac{C}{x-1}.$$

Since the denominators on the left and right are equal and the fractions are identically equal, then the numerators are equal too.

Let us reduce to the common denominator and equate the numerators:

$$15x^2-4x-81 = A(x+4)(x-1) + B(x-3)(x-1) + C(x-3)(x+4).$$

Let us consider *the second method of finding of undetermined coefficients.*

Let us substitute the first root of the denominator $x=3$ into this expression:

$$15\cdot 9 - 4\cdot 3 - 81 = A(3+4)(3-1) + B\cdot 0 + C\cdot 0,$$

$$42 = 14A \quad \text{or} \quad A = 3.$$

Let us substitute the second root of the denominator $x = -4$ into this expression:

$$15 \cdot 16 + 4 \cdot 4 - 81 = A \cdot 0 + B(-4-3)(-4-1) + C \cdot 0,$$
$$175 = 35B \quad \text{or} \quad B = 5.$$

Let us substitute the third root of the denominator $x = 1$ into this expression:

$$15 \cdot 1 - 4 \cdot 1 - 81 = A \cdot 0 + B \cdot 0 + C(1-3)(1+4),$$
$$-70 = -10C \quad \text{or} \quad C = 7.$$

So, $\int \frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} dx = \int \left(\frac{A}{x-3} + \frac{B}{x+4} + \frac{C}{x-1} \right) dx =$

$$= \int \left(\frac{3}{x-3} + \frac{5}{x+4} + \frac{7}{x-1} \right) dx = 3\int \frac{1}{x-3} dx + 5\int \frac{1}{x+4} dx + 7\int \frac{1}{x-1} dx =$$
$$= 3\ln|x-3| + 5\ln|x+4| + 7\ln|x-1| + C.$$

Integrating Trigonometric Functions. Integrals of Type $\int R(\sin x, \cos x) dx$

In this case, it is convenient to use the substitution $t = \operatorname{tg}\frac{x}{2}$, since the functions $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$ are rational functions of the variable x. The integration of the form $\int R(\sin x, \cos x) dx$ with the aid of the substitution $t = \operatorname{tg}\frac{x}{2}$ is sure to give the desired result, but it is because of its generality that this method may not be the best from the point of view of brevity and simplicity of the transformation involved.

For a number of special cases, simple change of variables are possible, which can be justified in the following way.

Let the expression $\int R(\sin x, \cos x)dx$ be even with respect to $\sin x$, then the given integral is reduced to an integral of a rational function since $\cos xdx = -d(\sin x) = -dt$ considering in a similar way, we arrive at a conclusion that in the case when the expression $\int R(\sin x, \cos x)dx$ is even with respect to $\cos x$, and the other is odd, the substitution $\cos x = t$ reduces the integral $\int R(\sin x, \cos x)dx$ to an integral of a rational function.

Let's now assume that the function $R(\sin x, \cos x)$ possesses one of the following properties:

a) both of them, $\sin x$ and $\cos x$, remain unchanged when $\sin x$ is replaced by $(-\sin x)$ and $\cos x$ by $(-\cos x)$;

b) both of them, $\sin x, \cos x$, change the sign when $\sin x$ is replaced by $(-\sin x)$ and $\cos x$ by $(-\cos x)$. It is sufficient to consider only the first case since the second case can be reduced to the first by multiplying both the numerator and denominator of the rational function $R(\sin x, \cos x)$ by $\sin x$ or $\cos x$. The integral $\int R(\sin x, \cos x)dx$ is reduced to an integral of a rational function by the substitution $t = tgx$ or $t = ctgx$ since $\cos^2 x = \frac{1}{1+t^2}$, $\sin^2 x = \frac{t^2}{1+t^2}$ and $dx = \frac{dt}{1+t^2}$ or $\cos^2 x = \frac{t^2}{1+t^2}$, $\sin^2 x = \frac{1}{1+t^2}$ and $dx = -\frac{dt}{1+t^2}$.

And now consider the integrals of the form

$$\int \sin^m x \cos^n xdx,$$

where m and n are integers. Then

a) if $m > 0$ is odd, $m = 2k - 1$ the substitution $\cos x = t$ immediately reduces the integral to a rational function;

b) if $n > 0$ is odd, $n = 2k - 1$ the substitution $\sin x = t$ yields the same result;

c) if both exponents m and n are positive and even, the result is readily obtained by means of trigonometric transformations with multiple arguments.

For instance, $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$. In conclusion we would like to note that integrals containing products of trigonometric functions of the form $\sin 2x$ and $\cos 2x$ are reduced to integrals of the form $\int \sin axdx$ and $\int \cos axdx$ by means of the following formulas:

$$\sin \alpha x \sin \beta x = \frac{1}{2}(\cos(\alpha - \beta)x - \cos(\alpha + \beta)x);$$

$$\sin \alpha x \cos \beta x = \frac{1}{2}(\sin(\alpha + \beta)x + \sin(\alpha - \beta)x);$$

$$\cos \alpha x \cos \beta x = \frac{1}{2}(\cos(\alpha + \beta)x + \cos(\alpha - \beta)x).$$

Example 19. Find the indefinite integral: $\int \sin^5 x \cos xdx$.

Solution. We have: $(\sin x)' = \cos x$, then $t = \sin x$ and $dt = \cos xdx$.

$$\int \sin^5 x \cos xdx \int t^5 dt = \frac{t^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

Example 20. Find the indefinite integral: $\int \sin^2 x \cos^2 xdx$.

Solution. We have:

$$\int \sin^2 x \cos^2 xdx = \int (\sin x \cos x)^2 dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx = \int \frac{1}{4}\sin^2 2xdx =$$

Let's use this formula $\sin^2 x = \frac{1-\cos 2x}{2}$ and get:

$$= \frac{1}{4}\int \sin^2 2xdx = \frac{1}{4}\int \frac{1-\cos 4x}{2}dx = \frac{1}{8}\int (1-\cos 4x)dx = \frac{1}{8}x - \frac{1}{8}\int \cos 4xdx.$$

The last integral is calculated with the help of the substitution $4x = t$, then $4dx = dt$. We get

$$\int \cos 4x dx = \int \cos t \frac{dt}{4} = \frac{1}{4}\sin t + C_1 = \frac{1}{4}\sin 4x + C_1.$$

Thus,

$$\int \sin^2 x \cos^2 x dx = \frac{1}{8}x - \frac{1}{8}\left(\frac{1}{4}\sin 4x + C_1\right) = \frac{1}{8}x - \frac{1}{32}\sin 4x + C,$$

where $-\frac{1}{8}C_1 = C$.

Example 21. Find the indefinite integral:

$$\int \sin 5x \cos 2x dx = \int \frac{1}{2}(\sin 7x + \sin 3x)dx = \frac{1}{2}\int \sin 7x dx + \frac{1}{2}\int \sin 3x dx =$$

$$= \frac{1}{2}\left(-\frac{\cos 7x}{7}\right) + \frac{1}{2}\left(-\frac{\cos 3x}{3}\right) = -\frac{\cos 7x}{14} - \frac{\cos 3x}{6} + C.$$

Example 22. Find the indefinite integral: $\int \frac{dx}{3 + 5\sin x + 3\cos x}$.

Solution. Let's use the substitution $\operatorname{tg}\frac{x}{2} = t$, then $\frac{x}{2} = \operatorname{arctg} t$, $dx = 2\frac{dt}{1+t^2}$ and we get

$$\int \frac{dx}{3 + 5\sin x + 3\cos x} = \int \frac{2}{3 + 5\cdot\frac{2t}{1+t^2} + 3\cdot\frac{1-t^2}{1+t^2}} \cdot \frac{dt}{1+t^2} =$$

$$= \int \frac{2(1+t^2)}{3(1+t^2) + 5\cdot 2t + 3(1-t^2)} \cdot \frac{dt}{1+t^2} = \int \frac{2dt}{3 + 3t^2 + 10t + 3 - 3t^2} =$$

$$= \int \frac{2dt}{10t+6} = \int \frac{dt}{5t+3} = \frac{1}{5}\ln|5t+3| + C = \frac{1}{5}\ln\left|5\operatorname{tg}\frac{x}{2} + 3\right| + C.$$

Integrals Containing Linear and Linear Fractional Irrationalities

1. Let's consider the indefinite integral of the form

$$\int R\left(x, x^{\frac{m}{n}}, ..., x^{\frac{r}{s}}\right)dx,$$

where *R* is a rational function of its arguments.

Let's use the substitution $x = t^k$, where *k* is a common denominator of the fractions $\frac{m}{n}, ..., \frac{r}{s}$, the integrand function is reduced to a rational function of t.

2. Let's consider the following integral:

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, ..., \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right)dx.$$

It is reduced to the integral of the rational function with the help of the substitution:

$$\frac{ax+b}{cx+d} = t^k,$$

where k is a common denominator of the fractions $\frac{m}{n}, ..., \frac{r}{s}$.

2. Let's consider the following integrals:

a) $\int R\left(x, \sqrt{a^2 - x^2}\right)dx,$

b) $\int R\left(x, \sqrt{a^2 + x^2}\right)dx,$

c) $\int R\left(x, \sqrt{x^2 - a^2}\right)dx.$

Such integrals are founded with the help of substitutions:

a) $x = a\sin t$ or $x = a\cos t$,
b) $x = a\text{tg}t$ or $x = a\text{ctg}t$,
c) $x = \dfrac{a}{\cos t}$ or $x = \dfrac{a}{\sin t}$.

Example 23. Find the indefinite integral: $\int \dfrac{\sqrt{(4-x^2)^3}}{x^6}dx$.

Solution. Let's use $x = 2\sin t$, then $dx = 2\cos t dt$ and $t = \arcsin\dfrac{x}{2}$.
Let's substitute and get:

$$\int \frac{\sqrt{(4-x^2)^3}}{x^6} = \int \frac{\sqrt{(4-4\sin^2 t)^3}}{64\sin^6 t}\cdot 2\cos t dt = \int \frac{8\sqrt{(1-\sin^2 t)^3}}{32\sin^6 t}\cos t dt =$$

$$= \frac{1}{4}\int \frac{\cos^3 t}{\sin^6 t}\cos t dt = \frac{1}{4}\int \frac{\cos^4 t}{\sin^6 t}dt = \frac{1}{4}\int \frac{\cos^4 t}{\sin^4 t}\cdot\frac{dt}{\sin^2 t} = \frac{1}{4}\int \text{ctg}^4 x\frac{dt}{\sin^2 t}.$$

Let's use the substitution $\text{ctg}t = z$, then $-\dfrac{1}{\sin^2 t}dt = dz$.

Thus,

$$\int \frac{\sqrt{(4-x^2)^3}}{x^6} = -\frac{1}{4}\int z^4 dz = -\frac{1}{4}\cdot\frac{z^5}{5} + C = -\frac{z^5}{20} + C = -\frac{\text{ctg}^5 t}{20} + C.$$

Using

$$\text{ctg}^5 x = \frac{\cos^5 t}{\sin^5 t} = \frac{\sqrt{(1-\sin^2 t)^5}}{\sin^5 t} = \frac{\sqrt{\left(1-\left(\frac{x}{2}\right)^2\right)^5}}{\left(\frac{x}{2}\right)^5} = \frac{\sqrt{\left(1-\frac{x^2}{4}\right)^5}}{\frac{x^5}{32}} = \frac{\sqrt{(4-x^2)^5}}{x^5},$$

we get

$$\int \frac{\sqrt{\left(4-x^2\right)^3}}{x^6}dx = -\frac{1}{20}\frac{\sqrt{\left(4-x^2\right)^5}}{x^5}+C.$$

Example 24. Find the indefinite integral: $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx.$

Solution. Let's find the common denominator of fractions $\frac{2}{3},\frac{1}{6},\frac{1}{3}$. It is 6, then we can use the substitution $x=t^6$ and $dx=6t^5dt$. Thus,

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx = \int \frac{t^6+t^4+t}{t^6\left(1+t^2\right)}\cdot 6t^5dt =$$

$$=6\int \frac{t\left(t^5+t^3+1\right)t^5}{t^6\left(1+t^2\right)}dt = 6\int \frac{t^5+t^3+1}{t^2+1}dt.$$

Let's get:

$$\begin{array}{r|l} -\,t^5+t^3+1 & \underline{t^2+1} \\ \underline{t^5+t^3} & t^3 \\ 1 & \end{array}.$$

Then

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx = 6\int\left(t^3+\frac{1}{t^2+1}\right)dt = 6\left(\frac{t^4}{4}+\operatorname{arctg}t\right)+C=$$

$$=6\left(\frac{\sqrt[6]{x^4}}{4}+\operatorname{arctg}\sqrt[6]{x}\right)+C=\frac{3}{2}\sqrt[3]{x^2}+6\operatorname{arctg}\sqrt[6]{x}+C.$$

Example 25. Find $\int \frac{\sqrt{x+4}}{x} dx$.

Solution. Let's use the substitution $x+4=t^2$, then $x=t^2-4$, and $dx=2tdt$. Thus

$$\int \frac{\sqrt{x+4}}{x} dx = \int \frac{t}{t^2-4} \cdot 2tdt = \int \frac{2t^2dt}{t^2-4} = 2\int \frac{t^2dt}{t^2-4}.$$

We get:

$$\begin{array}{r|l} -\;t^2 & \underline{t^2-4} \\ \underline{t^2-4} & 1 \\ 4 & \end{array}$$

and

$$\int \frac{\sqrt{x+4}}{x} dx = 2\int \left(1+\frac{4}{t^2-4}\right) dt = 2t + 8 \cdot \frac{1}{2 \cdot 2} ln \left|\frac{t-2}{t+2}\right| + C =$$
$$= 2\sqrt{x+4} + 2\,ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.$$

Applications of the Indefinite Integral in Business and Economics

Example 26. The monthly marginal cost function for a product is given as $MC(x) = x+25$. If 40 UAH are fixed costs, find the total cost function $TC(x)$ for the month.

Solution. We will use the cost function as the antiderivative of the marginal cost function:

$$TC(x) = \int MC(x)dx = \int (x+25)dx = \frac{x^2}{2} + 25x + C.$$

Let's use the fixed cost in order to find the value of the constant C. According to the condition of this task: $TC(0)=40$. We have

$$TC(0)=\frac{0^2}{2}+25\cdot 0+C=C=40 \qquad \text{or} \qquad C=40.$$

Thus, the total cost function for the month is $TC(x)=\frac{x^2}{2}+25x+40$.

Example 27. The marginal revenue function for a product is $MR(x)=88-10x$. The marginal cost function is $MC(x)=6x+40$, and the cost of producing 20 units is 2 280 UAH.

a) find the profit function $P(x)$;

b) find the profit or loss from selling 60 units.

Solution: a) let's remember the relationship between revenue and cost functions:

$$R(x)=P(x)+C(x).$$

Then

$$P(x)=R(x)-C(x).$$

We will use the revenue function and the cost function as the antiderivatives of the marginal revenue function and the marginal cost function correspondingly. Let's find them:

$$R(x)=\int MR(x)dx=\int(88-10x)dx=88x-10\frac{x^2}{2}+C_1=88x-5x^2+C_1,$$

Let's use $R(0)=0$. It gives us that $C_1=0$, so $R(x)=88x-5x^2$.

Let's find

$$C(x)=\int MC(x)dx=\int(6x+40)dx=6\frac{x^2}{2}+40x+C_1=3x^2+40x+C_2$$

and

$$C(20)=2\,280=3\cdot 20^2+40\cdot 20+C_2$$

or $C_2 = 2280 - 3\cdot 20^2 - 40\cdot 20 = 2\,280 - 1\,200 - 800 = 280$.

Then

$$C(x) = 3x^2 + 40x + 280.$$

Let's find the profit function $P(x)$:

$$P(x) = R(x) - C(x) = 88x - 5x^2 - \left(3x^2 + 40x + 280\right) = 48x - 8x^2 - 280;$$

b) find the profit or loss from selling 60 units:

$$P(60) = 48\cdot 60 - 8\cdot 60^2 - 280 = 2\,880 - 28\,800 - 280 = -26\,200,$$

so there is a loss of 26 200 UAH.

Example 28. The average cost of a product changes at the rate $\overline{C}'(x) = -12x^{-2} + \frac{1}{12}$. The average cost of 12 units is 20. Find the average cost of producing 24 units.

Solution. First, we need the average cost function, then we can evaluate it at $x = 24$:

$$\overline{C}(x) = \int \overline{C}'(x)dx = \int\left(-12x^{-2} + \frac{1}{12}\right)dx = -\frac{12x^{-2+1}}{-2+1} + \frac{1}{12}x + C = \frac{12}{x} + \frac{x}{12} + C.$$

Let's use $\overline{C}(12) = 20$, then $20 = \frac{12}{12} + \frac{12}{12} + C$ or $C = 20 - 1 - 1 = 18$.

Thus, $\overline{C}(x) = \frac{12}{x} + \frac{x}{12} + 18$.

Let's find: $\overline{C}(24) = \frac{12}{24} + \frac{24}{12} + 18 = \frac{1}{2} + 2 + 18 = 20.5$.

Individual tasks

Find indefinite integrals:

Variant 1

1. $\int \frac{\left(\sqrt{x}+2\right)^2}{\sqrt[4]{x}}dx.$ 2. $\int \frac{3x+1}{3x-2}dx.$ 3. $\int (4-3x)e^{-3x}dx.$

4. $\int \left(x^2+6\right)\cos 2x dx.$ 5. $\int \frac{dx}{x^2+2x+5}.$ 6. $\int \frac{x^3+1}{x^2+x}dx.$

7. $\int \frac{6x^2+13x+9}{(x+1)(x+2)^2}dx.$ 8. $\int \frac{4x^2+4x+2}{(x+1)\left(x^2+x+1\right)}dx.$

Variant 2

1. $\int \left(\sqrt[3]{x}+1\right)\left(x-\sqrt{x}+7\right)dx.$ 2. $\int 4^{2x-1}dx.$ 3. $\int (3x+4)e^{3x}dx.$

4. $\int \frac{xdx}{x^2-7x+13}.$ 5. $\int \left(x^2-4\right)\cos 5x dx.$ 6. $\int \frac{x^3-17}{x^2-4x+3}dx.$

7. $\int \frac{6x^2+13x+8}{x(x+2)^2}dx.$ 8. $\int \frac{4x^2+3x+2}{(x+1)\left(x^2+1\right)}dx.$

Variant 3

1. $\int \frac{\left(\sqrt[3]{x}+1\right)^2}{\sqrt{x}}dx.$ 2. $\int \frac{dx}{\sqrt{7+8x^2}}.$ 3. $\int (4-16x)\sin 4x dx.$

4. $\int \left(x^2+4x+3\right)\cos x dx.$ 5. $\int \frac{dx}{x^2+4x+8}.$ 6. $\int \frac{3x^3+1}{x^2-1}dx.$

7. $\int \frac{-6x^2+13x-6}{(x+2)(x-2)^2}dx.$ 8. $\int \frac{7x^2+7x-1}{(x+2)\left(x^2+x+1\right)}dx.$

Variant 4

1. $\int \frac{\left(\frac{1}{x}-5\right)^2}{\sqrt{x}}dx.$ 2. $\int \frac{\ln x}{x}dx.$ 3. $\int (1-6x)e^{2x}dx.$

4. $\int (x+2)^2 \cos 7x dx$. 5. $\int \frac{7-8x}{2x^2-3x+1} dx$. 6. $\int \frac{2x^3+5}{x^2-x-2} dx$.

7. $\int \frac{6x^2+14x+10}{(x-1)(x+2)^2} dx$. 8. $\int \frac{4x^2+2x-1}{(x+1)(x^2+2x+2)} dx$.

Variant 5

1. $\int \frac{(\sqrt{x}-1)^3}{x} dx$. 2. $\int x \cdot 7^{x^2} dx$. 3. $\int \ln(4x^2+1) dx$.

4. $\int (x^2+7x+12)\cos 6x dx$. 5. $\int \frac{dx}{3x^2-x+1}$. 6. $\int \frac{2x^3-1}{x^2+x-6} dx$.

7. $\int \frac{-6x^2+11x-10}{(x-2)(x+2)^2} dx$. 8. $\int \frac{6x^2+9x+6}{(x+1)(x^2+2x+3)} dx$.

Variant 6

1. $\int \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt[4]{x^3}} \right) dx$. 2. $\int \frac{dx}{\sin^2 \frac{x}{7}}$. 3. $\int \operatorname{arctg}\sqrt{6x-1} dx$.

4. $\int (2x^2+4x-7)\cos 2x dx$. 5. $\int \frac{3x-6}{\sqrt{x^2-4x+5}} dx$. 6. $\int \frac{3x^3+25}{x^2+3x+2} dx$.

7. $\int \frac{6x^2+11x+7}{(x+1)(x+2)^2} dx$. 8. $\int \frac{11x^2+16x+10}{(x+2)(x^2+2x+3)} dx$.

Variant 7

1. $\int \frac{(\sqrt[3]{2}+\sqrt{x})^2}{\sqrt{3x}} dx$. 2. $\int \cos\sqrt{x} \cdot \frac{dx}{\sqrt{x}}$. 3. $\int e^{-3x}(2-9x) dx$.

4. $\int (9x^2+9x+11)\cos 8x dx$. 5. $\int \frac{x dx}{\sqrt{5x^2-2x+1}}$. 6. $\int \frac{6x^2+5x-1}{(x+1)(x^2+2)} dx$.

7. $\int \frac{3x^3+2x^2+1}{(x+2)(x-2)(x-1)} dx$. 8. $\int \frac{6x^2+7x+1}{(x-1)(x+1)^2} dx$.

Variant 8

1. $\int \frac{(1-x)^2}{x\sqrt{x}}dx.$ 2. $\int x \cdot \sin(1-x^2)dx.$ 3. $\int (5x+6)\cos 2x dx.$

4. $\int (8x^2+16x+17)\cos 4x dx.$ 5. $\int \frac{dx}{x^2+4x+9}.$ 6. $\int \frac{6x^2+10x+10}{(x-1)(x+2)^2}dx.$

7. $\int \frac{3x^3+2x^2+1}{(x-1)(x-2)(x-3)}dx.$ 8. $\int \frac{2x^2+2x+1}{(x+1)(x^2+2x+2)}dx.$

Variant 9

1. $\int \left(\frac{1}{x}-2\sqrt{x}\right)^2 dx.$ 2. $\int 2^{\sin x} \cdot \cos x dx.$ 3. $\int (\sqrt{2} \cdot x-3)\cos 2x dx.$

4. $\int (3-7x^2)\cos 9x dx.$ 5. $\int \frac{3x-2}{x^2+6x+9}dx.$ 6. $\int \frac{x^3}{(x-1)(x+1)(x+2)}dx.$

7. $\int \frac{6x^2+7x+2}{x(x+1)^2}dx.$ 8. $\int \frac{9x^2+21x+21}{(x+3)(x^2+3)}dx.$

Variant 10

1. $\int \frac{(1-x)^2}{x\sqrt[3]{x}}dx.$ 2. $\int \frac{x^3 dx}{\cos^2 x^4}.$ 3. $\int (2x-5)\cos 4x dx.$

4. $\int (3x+5)^2 \sin x dx.$ 5. $\int \frac{dx}{\sqrt{x^2+6x+8}}.$ 6. $\int \frac{x^3-3x^2-12}{(x-1)(x-3)(x-2)}dx.$

7. $\int \frac{-6x^2+13x+8}{x(x-2)^2}dx.$ 8. $\int \frac{x^2+8x+8}{(x+2)(x^2+4)}dx.$

Variant 11

1. $\int \left(\frac{1}{\sqrt[3]{x}}+\frac{2}{\sqrt{x}}\right)^2 dx.$ 2. $\int e^{\sin^2 x} \cdot \sin 2x dx.$ 3. $\int (x+5)\sin 3x dx.$

4. $\int (3-7x^2)\cos 2x dx.$ 5. $\int \frac{x+1}{x^2-4x+3}dx.$ 6. $\int \frac{x^3-3x^2-12}{(x-4)(x-3)x}dx.$

7. $\int \frac{-6x^2+13x-7}{(x+1)(x-2)^2}dx.$ 8. $\int \frac{5x^2+12x+4}{(x+2)(x^2+1)}dx.$

Variant 12

1. $\int (\sqrt[3]{x}-1)(\sqrt{x}+\sqrt[4]{x})dx.$ 2. $\int 5^{3x+2}dx.$ 3. $\int \operatorname{arctg}\sqrt{4x-1}dx.$

4. $\int (1-8x^2)\cos 7x dx.$ 5. $\int \frac{dx}{\sqrt{2+3x-2x^2}}.$ 6. $\int \frac{4x^3+x^2+2}{x(x-1)(x+2)}dx.$

7. $\int \frac{-6x^2+14x-6}{(x+1)(x-2)^2}dx.$ 8. $\int \frac{-4x^2-16x-12}{(x-1)(x^2+4x+5)}dx.$

Variant 13

1. $\int \frac{2x^2+x-1}{x^3}dx.$ 2. $\int \frac{\arcsin x}{\sqrt{1-x^2}}dx.$ 3. $\int (4x-2)\cos 2x dx.$

4. $\int (x^2-3x)\sin 5x dx.$ 5. $\int \frac{2x-8}{\sqrt{1-x-x^2}}dx.$ 6. $\int \frac{3x^3-2}{x^3-x}dx.$

7. $\int \frac{-6x^2+10x-10}{(x+1)(x-2)^2}dx.$ 8. $\int \frac{13x^2-13x+1}{(x-2)(x^2-x+1)}dx.$

Variant 14

1. $\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}dx.$ 2. $\int \frac{dx}{x\ln^3 x}.$ 3. $\int (5x-2)e^{3x}dx.$

4. $\int (x^2+2x+1)\sin 4x dx.$ 5. $\int \frac{xdx}{x^2-4x+7}.$ 6. $\int \frac{x^3-3x^2-12}{(x-4)(x-2)x}dx.$

7. $\int \frac{6x^2+2x+3}{(x+2)x^2}dx.$ 8. $\int \frac{7x^2+x-46}{(x-1)(x^2+9)}dx.$

Variant 15

1. $\int \frac{x^2+5x-1}{\sqrt[3]{x}}dx.$ 2. $\int e^{-\left(x^2+1\right)}\cdot xdx.$ 3. $\int \ln\left(x^2-3\right)dx.$

4. $\int \left(x^2-3x+2\right)\sin xdx.$ 5. $\int \frac{2x-1}{x^2+2x+3}dx.$ 6. $\int \frac{x^5-x^3+1}{x^2-x}dx.$

7. $\int \frac{9x^2+10x+2}{(x-1)(x+2)^2}dx.$ 8. $\int \frac{24x^2+20x-28}{(x+3)\left(x^2+2x+2\right)}dx.$

Variant 16

1. $\int \frac{5x^8+1}{x^4}dx.$ 2. $\int \frac{dx}{3x^2+5}.$ 3. $\int (2-x)\sin 2xdx.$

4. $\int \left(x^2-5x+6\right)\cos xdx.$ 5. $\int \frac{x-1}{x^2+4x+7}dx.$ 6. $\int \frac{x^5+3x^3-1}{x^2+x}dx.$

7. $\int \frac{2x^2+x+1}{(x+1)x^2}dx.$ 8. $\int \frac{3x^2+3x+2}{(x+5)\left(x^2+1\right)}dx.$

Variant 17

1. $\int \frac{x-1}{\sqrt[5]{x^4}}dx.$ 2. $\int \left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)dx.$ 3. $\int e^{-2x}(4x-3)dx.$

4. $\int \left(x^2+6x+9\right)\sin 5xdx.$ 5. $\int \frac{xdx}{\sqrt{x^2+x+1}}.$ 6. $\int \frac{2x^5-8x^3+3}{x^2-2x}dx.$

7. $\int \frac{6x^2+7x+4}{(x+2)(x+1)^2}dx.$ 8. $\int \frac{11x^2+21x+21}{(x-4)\left(x^2+9\right)}dx.$

Variant 18

1. $\int \frac{\left(\sqrt{2}-\sqrt{x}\right)^3}{\sqrt{2x}}dx.$ 2. $\int \frac{\operatorname{arctg}\frac{x}{2}}{4+x^2}dx.$ 3. $\int \operatorname{arcctg}\sqrt{2x-1}dx.$

4. $\int\left(1-5x^2\right)\sin 3x dx.$ 5. $\int\frac{1-2x}{\sqrt{x^2+2x+5}}dx.$ 6. $\int\frac{3x^5-12x^3-7}{x^2+2x}dx.$

7. $\int\frac{6x^2+5x}{(x+2)(x-3)^2}dx.$ 8. $\int\frac{x^2+x+3}{(x-6)\left(x^2+x+1\right)}dx.$

Variant 19

1. $\int\frac{\left(x^2+2\right)\left(x^2-1\right)}{\sqrt[4]{x^3}}dx.$ 2. $\int\frac{xdx}{x^2-4}.$ 3. $\int\operatorname{arctg}\sqrt{5x-1}dx.$

4. $\int\left(x^2+35\right)\sin 8x dx.$ 5. $\int\frac{x-3}{\sqrt{x^2+2x+6}}dx.$ 6. $\int\frac{-x^5+9x^3+4}{x^2+3x}dx.$

7. $\int\frac{6x^2+7x}{(x-2)(x+1)^2}dx.$ 8. $\int\frac{4x^2+2x+2}{(x-1)\left(x^2+2\right)}dx.$

Variant 20

1. $\int\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^3 dx.$ 2. $\int\frac{2x+5}{2x-1}dx.$ 3. $\int(3x-2)\cos 5x dx.$

4. $\int\left(3x-x^2\right)\sin 4x dx.$ 5. $\int\frac{3x+2}{2x^2+x+5}dx.$ 6. $\int\frac{x^3-5x^2+5x+23}{x^2+5x}dx.$

7. $\int\frac{6x^2+5x+4}{(x-2)(x-3)}dx.$ 8. $\int\frac{7x^2+7x+9}{(x+1)\left(x^2+x+2\right)}dx.$

Variant 21

1. $\int\left(\frac{1-x}{x}\right)^2 dx.$ 2. $\int\frac{\sqrt{x}+\ln x}{x}dx.$ 3. $\int(4x+7)\cos 3x dx.$

4. $\int(x+1)\ln^2(x+1)dx.$ 5. $\int\frac{2x-1}{\sqrt{2x^2+3x-1}}dx.$ 6. $\int\frac{4x^2+4x+3}{(x-1)\left(x^2+1\right)}dx.$

7. $\int\frac{-2x^3+5x^2-7x+9}{(x+3)(x-1)x}dx.$ 8. $\int\frac{6x^2+4x+24}{(x-1)(x+2)^2}dx.$

Variant 22

1. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx.$ 2. $\int\frac{x+3}{\sqrt{x^2+4}}dx.$ 3. $\int(8-3x)\cos 5x dx.$

4. $\int x\ln^2 x dx.$ 5. $\int\frac{xdx}{\sqrt{1-x-x^2}}.$ 6. $\int\frac{-x^5+25x^3+1}{x^2+5x}dx.$

7. $\int\frac{6x^2+14x+4}{(x+1)(x-1)^2}dx.$ 8. $\int\frac{3x^2+2x+6}{(x+2)(x^2+3)}dx.$

Variant 23

1. $\int\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt[4]{x^3}}\right)x dx.$ 2. $\int(2-9x)^9 dx.$ 3. $\int(2-3x)\sin 2x dx.$

4. $\int\frac{\ln^2 x}{\sqrt{x}}dx.$ 5. $\int\frac{xdx}{3x^2+2x+8}.$ 6. $\int\frac{2x^3-8}{x(x-2)(x+2)}dx.$

7. $\int\frac{6x^2+18x-4}{(x-3)(x+1)^2}dx.$ 8. $\int\frac{2x^2-x+1}{(x-3)(x^2+1)}dx.$

Variant 24

1. $\int\frac{(\sqrt[3]{x}+\sqrt[4]{x})(\sqrt{x}+1)}{x}dx.$ 2. $\int\frac{\sqrt{\operatorname{tg}x}}{\cos^2 x}dx.$ 3. $\int(3x+4)\sin 5x dx.$

4. $\int\frac{\ln^2 x}{\sqrt[3]{x^2}}dx.$ 5. $\int\frac{5x+2}{x^2+4x+6}dx.$ 6. $\int\frac{4x^3+2x^2-x-3}{x(x-1)(x+1)}dx.$

7. $\int\frac{-6x^2+14x-4}{(x+5)x^2}dx.$ 8. $\int\frac{x^2+2x-2}{(x+6)(x^2-x+1)}dx.$

Variant 25

1. $\int\left(\frac{1}{2\sqrt{x}}+1\right)^2 x^2 dx.$ 2. $\int\frac{5}{tg\frac{x}{5}}dx.$ 3. $\int(\sqrt{2}-8x)\sin 3x dx.$

4. $\int e^{2x}\cdot \sin x dx.$ 5. $\int \frac{dx}{x^2+6x+11}.$ 6. $\int \frac{3x^3-5x^2+2}{x(x-1)(x+2)}dx.$

7. $\int \frac{6x^2+10x+12}{(x+3)x^2}dx.$ 8. $\int \frac{5x^2+5x-7}{(x-6)(x^2+4)}dx.$

Variant 26

1. $\int \frac{(1-\sqrt{x})(x+1)}{\sqrt{x}}dx.$ 2. $\int \sin^2 x dx.$ 3. $\int (2x-3)2^{1-2x}dx.$

4. $\int e^{-3x}\cos x dx.$ 5. $\int \frac{xdx}{\sqrt{x^2+4x-5}}.$ 6. $\int \frac{2x^3+20}{x(x-4)(x+5)}dx.$

7. $\int \frac{6x^2+15x+2}{(x-4)(x-1)^2}dx.$ 8. $\int \frac{3x^2+x-4}{(x+1)(x^2+3x+4)}dx.$

Variant 27

1. $\int \left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)^3 dx.$ 2. $\int \cos^2\frac{x}{2}dx.$ 3. $\int (2-x)5^{6x}dx.$

4. $\int e^{x}\cos 2x dx.$ 5. $\int \frac{1-2x}{5x^2+5x+4}dx.$ 6. $\int \frac{-6x^3+13x+6}{x(x-3)(x+2)}dx.$

7. $\int \frac{-6x^2+7x-4}{(x-1)^2\cdot(x+4)}dx.$ 8. $\int \frac{2x^2+x+1}{x(x^2+2x+2)}dx.$

Variant 28

1. $\int \frac{(x^3+3x)^2}{\sqrt{x}}dx.$ 2. $\int \frac{3dx}{\sqrt{5x^2-7}}.$ 3. $\int \left(1-\frac{x}{3}\right)3^{x+2}dx.$

4. $\int e^{5x}\sin x dx.$ 5. $\int \frac{x+3}{1-x-3x^2}dx.$ 6. $\int \frac{3x^3-x^2-12x-2}{x(x+1)(x+2)}dx.$

7. $\int \frac{-6x^2+7x}{(x+2)(x-1)^2}dx.$ 8. $\int \frac{x+4}{(x+2)(x^2+2)}dx.$

Variant 29

1. $\int \frac{\left(\frac{1}{2\sqrt{x}}+1\right)^3}{x}dx.$ 2. $\int \frac{4x^3}{x^8+5}dx.$ 3. $\int (x+1)\sin(x+1)dx.$

4. $\int \sqrt{x}\ln^2 x dx.$ 5. $\int \frac{4x-8}{\sqrt{5x^2-2x+1}}dx.$ 6. $\int \frac{2x^3-9}{x(x-1)(x+3)}dx.$

7. $\int \frac{6x^2-10x+52}{(x-2)x^2}dx.$ 8. $\int \frac{3x^2+3x+2}{(x+3)(x^2+2x+3)}dx.$

Variant 30

1. $\int \frac{\left(x+\frac{1}{x}\right)^2}{2\sqrt[3]{x}}dx.$ 2. $\int tg\sqrt[3]{x}\cdot\frac{dx}{\sqrt[3]{x^2}}.$ 3. $\int \frac{x}{2}\cdot 9^{2-x}dx.$

4. $\int x^2e^{5x}dx.$ 5. $\int \frac{5x-10}{x^2-7x+13}dx.$ 6. $\int \frac{2x^3-7x}{x(x-3)(x+1)}dx.$

7. $\int \frac{-6x^2+13x-6}{(x-5)x^2}dx.$ 8. $\int \frac{7x^2+12x+6}{x(x^2+1)}dx.$

Theoretical Questions

1. An antiderivative.
2. An indefinite integral.
3. The integral sign.
4. The integrand expression.
5. The integration variable.
6. An arbitrary constant.
7. The key theorem on antidertivatives.
8. Properties of the indefinite integral.
9. The theorem of invariance of the integration formula
10. The basic table of integrals.
11. Change of the variable (substitution) in the indefinite integral.
12. Different substitutions.
13. Integration by parts.
14. Three classes (types) of integrals.
15. Integration of rational functions.
16. A polynomial of the n-th degree.
17. Proper and improper integrals.
18. Linear and quadratic factors.
19. Decomposition of a fraction into partial fractions.
20. The method of indeterminate coefficients.
21. Integrating trigonometric functions.
22. Integrals containing linear and linear fractional irrationalities.
23. Integrals containing quadratic irrationalities.
24. Integrating some simple irrational functions.

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НАВЧАЛЬНЕ ВИДАННЯ

Методичні рекомендації до виконання практичних завдань з невизначеного інтегралу з навчальної дисципліни "МАТЕМАТИЧНИЙ АНАЛІЗ ТА ЛІНІЙНА АЛГЕБРА"

для іноземних студентів та студентів, що навчаються англійською мовою, напряму підготовки 6.030601 "Менеджмент" спеціалізації "Бізнес-адміністрування" денної форми навчання (англ. мовою)

Укладач **Місюра** Євгенія Юріївна

Відповідальний за видання *Л. М. Малярець*

Редактор *З. В. Зобова*

Коректор *З. В. Зобова*

Викладено необхідний теоретичний матеріал з навчальної дисципліни та наведено типові приклади, які сприяють найбільш повному засвоєнню матеріалу з теми "Невизначений інтеграл" і застосуванню отриманих знань на практиці. Подано завдання для індивідуальної роботи та перелік теоретичних питань, що сприяють удосконаленню та поглибленню знань студентів з даної теми.

Рекомендовано для студентів напряму підготовки 6.030601 "Менеджмент" денної форми навчання.

План 2016 р. Поз. № 37.

Підп. до друку 07.10.2016 р. Формат 60 × 90 1/16. Папір офсетний. Друк цифровий.

Ум. друк. арк. 3,0. Обл.-вид. арк. 3,75. Тираж 50 пр. Зам. № 178.

Видавець і виготовлювач – ХНЕУ ім. С. Кузнеця, 61166, м. Харків, просп. Науки, 9-А

Свідоцтво про внесення суб'єкта видавничої справи до Державного реєстру ***ДК № 4853 від 20.02.2015 р.***