MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

SIMON KUZNETS KHARKIV NATIONAL UNIVERSITY OF ECONOMICS

Guidelines for laboratory work
on the academic discipline

"MATHEMATICAL MODELLING IN ECONOMICS AND MANAGEMENT: OPERATIONS RESEARCH"

for full-time students of training direction
6.030601 "Management"

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Самостійне електронне текстове мережеве видання

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G 94 **Guidelines** for laboratory work on the academic discipline "Mathematical Modelling in Economics and Management: Operations Research" for full-time students of training direction 6.030601 "Management" : [Electronic edition] / compiled by N. Chernova, O. Polyakova. – Kharkiv : S. Kuznets KhNUE, 2016. – 67 p. (English)

Guidelines for laboratory work are presented according to the syllabus of the academic discipline. Examples and tasks for self-study are provided for each theme.

For full-time students of training direction 6.030601 "Management".

Introduction

Mathematical Modelling in Economics and Management: Operations Research (OR) is the application of scientific methods to the management and administration of organized governmental, commercial, and industrial processes. It uses mathematical techniques to solve management problems.

Operations Research is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources. A system is an organization of interdependent components that work together to accomplish the goal of the system.

The principal phases for implementing OR in practice include the definition of the problem, model construction, model solution, model validation, implementation of the solution.

Six guidelines are presented below according to the syllabus of the academic discipline. All the guidelines consist of theoretical material and readymade examples of some operations research problems within the appropriate theme. Then tasks for self-study activity are given.

All OR problems are solved using Solver Add-in. The Solver Add-in is a Microsoft Office Excel add-in program that is available when you install Microsoft Office or Excel.

To use the Solver Add-in, however, you first need to load it in Excel.

1. Click the **Microsoft Office Button**, and then click **Excel Options**.
2. Under **Add-ins**, select **Solver Add-in** and click on the **Go** button (Fig. 1).
3. In the Add-Ins available box, click Solver Add-in and click OK (Fig. 2).

If Solver Add-in is not listed in the Add-Ins available box, click **Browse** to locate the add-in. If you get prompted that the Solver Add-in is not currently installed on your computer, click **Yes** to install it.

4. After you have loaded the Solver Add-in, the **Solver** command is available in the **Analysis** group on the **Data** tab (Fig. 3).

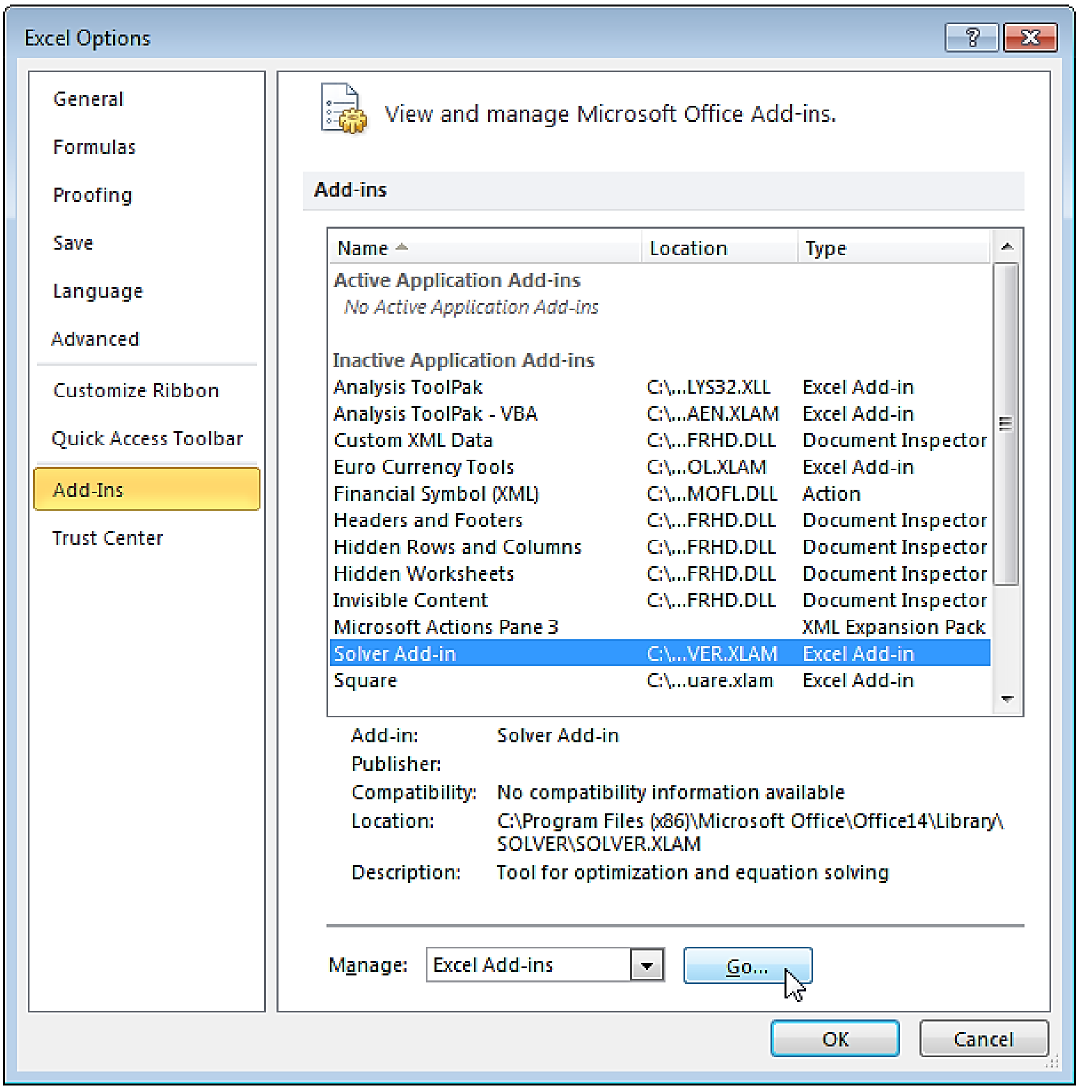

Fig. 1. **The Excel Options**

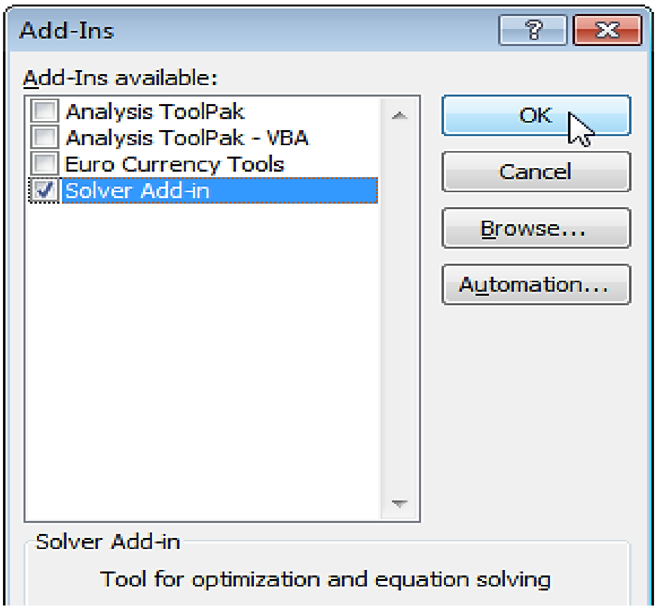

Fig. 2. **The Add-Ins**

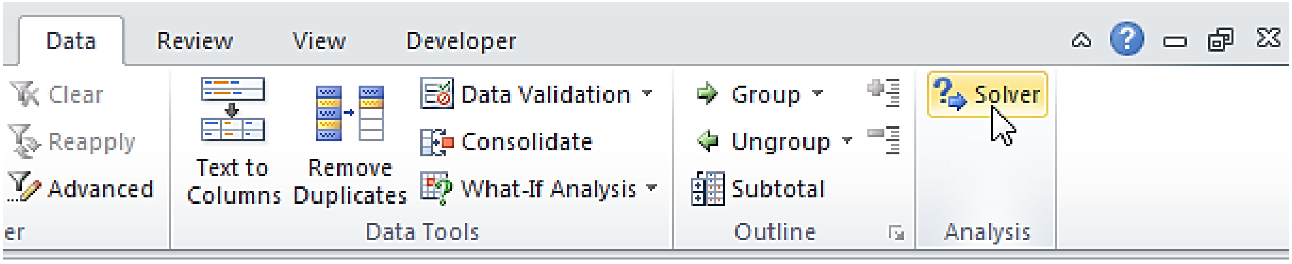

Fig. 3. **The Solver**

Guidelines and tasks for self-study work

Theme 2. The Product Mix Problem

The linear programming (LP) model, as any OR model, has three basic components:

1. Decision variables that we seek to determine.
2. The objective (goal) that we need to optimize (maximize or minimize).
3. Constraints that the solution must satisfy.

The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.

Example 1. Reddy Mikks produces both pink and yellow paints from two raw materials, M1 and M2. Table 1 provides the basic data of the problem.

Table 1

The initial data

	Tons of raw material per ton of		Maximum daily availability (tons)
	yellow paint	pink paint	
Raw material M1	6	4	24
Raw material M2	1	2	6
Profit per ton ($1000)	5	4	

Reddy Mikks wants to determine the optimum (best) product mix of yellow and pink paints that maximizes the total daily profit.

For the Reddy Mikks problem, we need to determine the daily amounts of each paint to be produced.

Thus, the variables of the model are defined as

x_1– yellow paint produced daily (tons);

x_1– pink paint produced daily (tons).

These variables are our unknowns, for which we need to find suitable values.

To construct the objective function, note that the company wants to maximize (i.e., increase as much as possible) the total daily profit of both paints. Given that the profits per ton of yellow and pink paints are 5 and 4 thousand dollars respectively, it follows that

total profit from the yellow paint equals $5x_1$ thousand dollars;

total profit from the pink paint equals $4x_2$ thousand dollars.

Letting F represent the total daily profit, the objective of the company is

$$F = 5x_1 + 4x_2.$$

Next, we construct the constraints that restrict the raw material usage and product demand. The raw material restrictions are expressed verbally as:

usage of a raw material by both paints should be less or equal maximum raw material availability.

We can calculate the usage of the raw material M1 by the yellow paint as:

$$6x_1 \text{ tons/day.}$$

We can calculate the usage of the raw material M1 by the pink paint as:

$$4x_2 \text{ tons/day.}$$

Hence the usage of the raw material M1 by both paints equals

$$6x_1 + 4x_2 \text{ tons/day.}$$

In a similar manner, the usage of the raw material M2 by both paints equals

$$1x_1 + 2x_2 \text{ tons/day.}$$

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons respectively, the associated restrictions are given as

$$6x_1 + 4x_2 \le 24 \text{ (raw material MI)}$$

$$1x_1 + 2x_2 \le 6 \text{ (raw material M2)}$$

A hidden (or "understood-to-be") restriction is that variables x_1 and x_2 cannot assume negative values:

$$x_1 \ge 0,\ x_2 \ge 0.$$

The complete Reddy Mikks model is

$$F = 5x_1 + 4x_2 \to \max$$

$$\begin{cases} 6x_1 + 4x_2 \le 24 \\ 1x_1 + 2x_2 \le 6 \\ x_1 \ge 0 \\ x_2 \ge 0 \end{cases}$$

The model we are going to solve looks as follows in Excel (Fig. 4).

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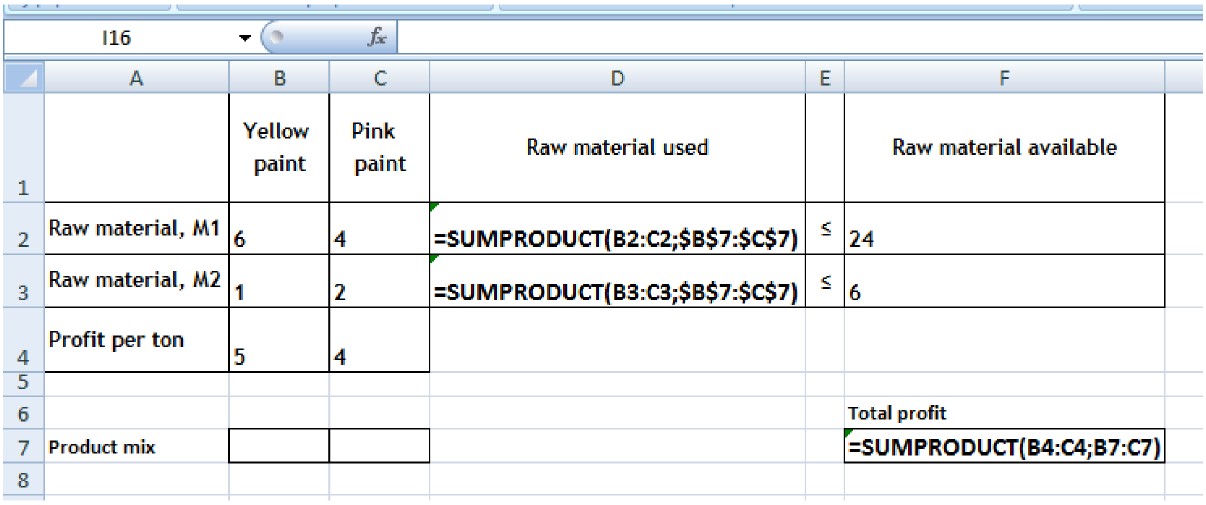
	A	B	C	D	E	F
1		Yellow paint	Pink paint	Raw material used		Raw material available
2	Raw material, M1	6	4		≤	24
3	Raw material, M2	1	2		≤	6
4	Profit per ton	5	4			
5						
6						Total profit
7	Product mix					
8						

Fig. 4. **The initial data**

Insert the following three SUMPRODUCT functions (Fig. 5).



	A	B	C	D	E	F
1		Yellow paint	Pink paint	Raw material used		Raw material available
2	Raw material, M1	6	4	=SUMPRODUCT(B2:C2;$B$7:$C$7)	≤	24
3	Raw material, M2	1	2	=SUMPRODUCT(B3:C3;$B$7:$C$7)	≤	6
4	Profit per ton	5	4			
5						
6						Total profit
7	Product mix					=SUMPRODUCT(B4:C4;B7:C7)
8						

Fig. 5. **The functions to be inserted**

To find the optimal solution, execute the following steps.

1. On the Data tab, click Solver (see Fig. 6).

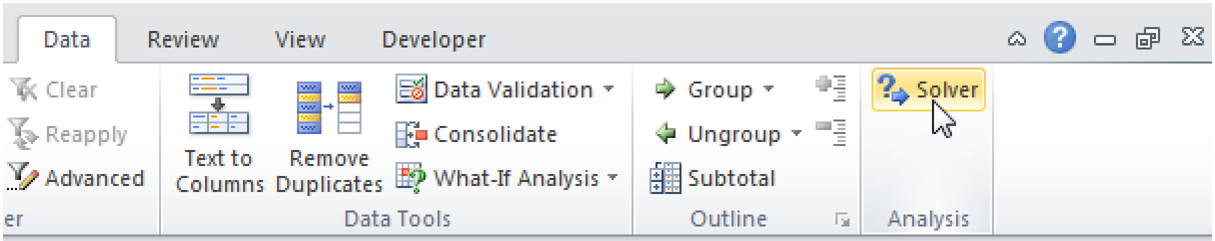


Fig. 6. **The Solver**

Enter the solver parameters. The result should be consistent with Fig. 7.

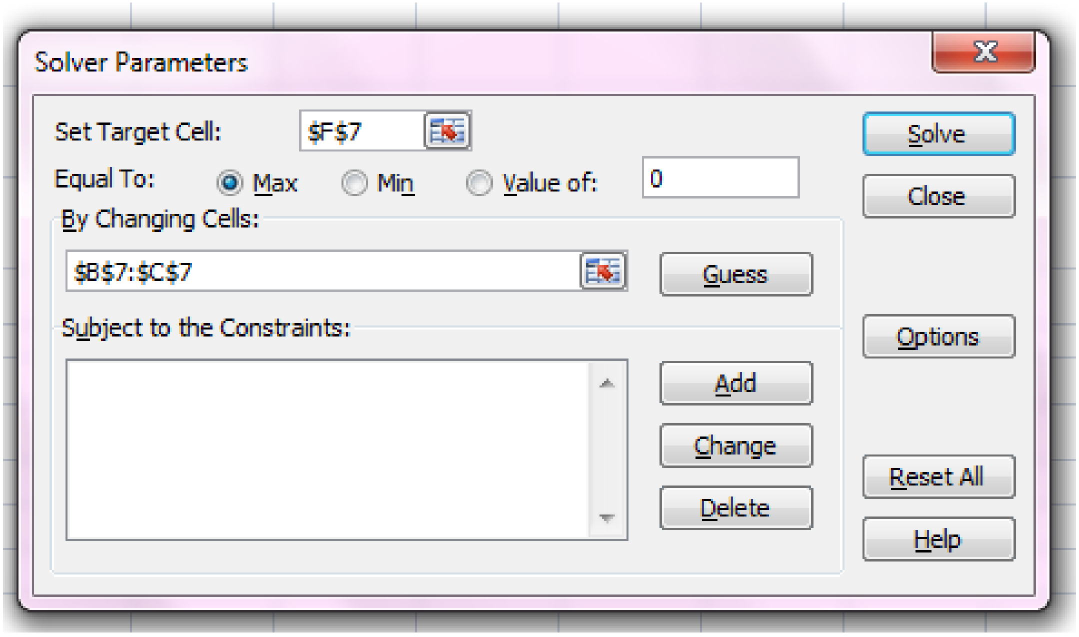


Fig. 7. **The Solver parameters**

2. Enter F7 for the Objective.
3. Click Max.
4. Enter B7:C7 for the Changing Variable Cells.
5. Click Add to enter the constraints (see Fig. 8).

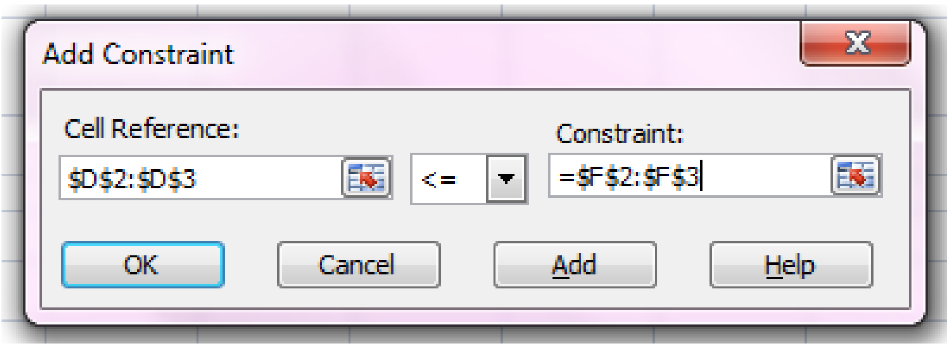


Fig. 8. **The Add constraint**

6. Click Options and select Assume Linear Model and Assume Non-Negative (Fig. 9).

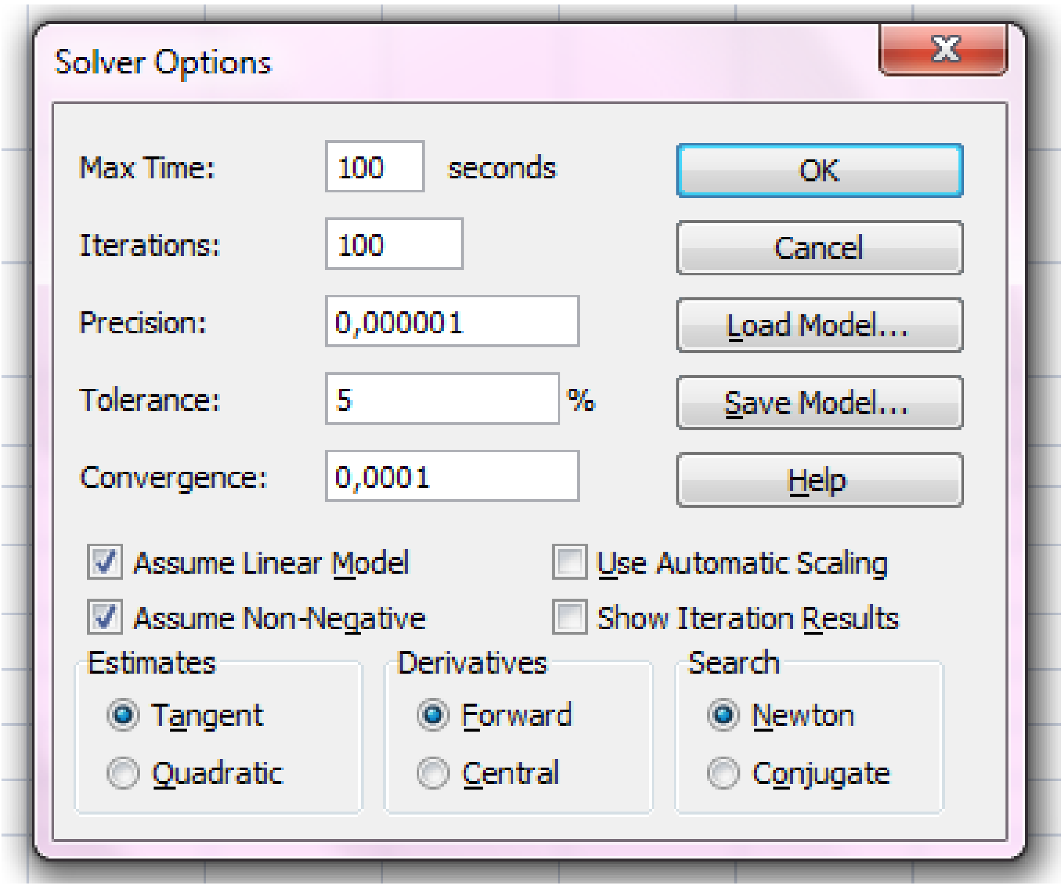


Fig. 9. **The Solver options**

7. Finally, click Solve. The result is shown in Fig. 10. Choose Answer and Sensitivity.

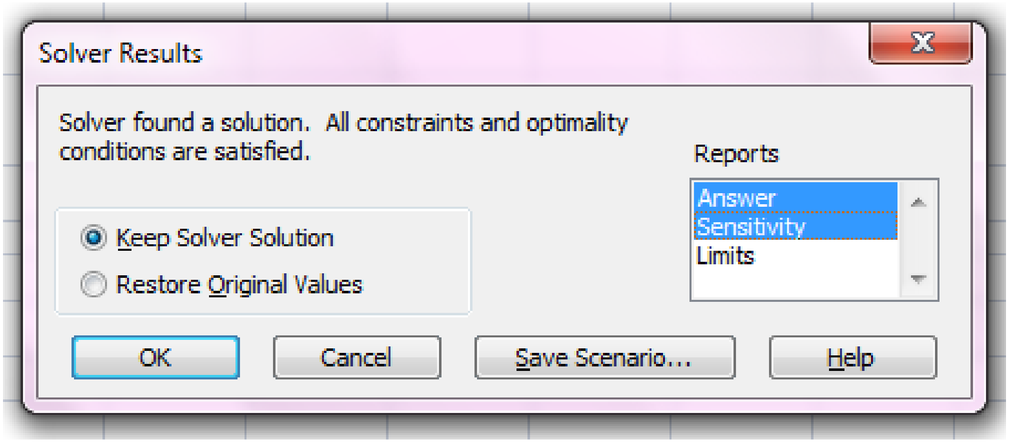


Fig. 10. **The Solver results**

The optimal solution is shown in Fig. 11.

Conclusion: it is optimal to produce 3 tons of yellow paint and 1.5 tons of pink paint. This solution gives the maximum profit of 21. This solution uses all the resources available.

	A	B	C	D	E	F
1		Yellow paint	Pink paint	Raw material used		Raw material available
2	Raw material, M1	6	4	24	≤	24
3	Raw material, M2	1	2	6	≤	6
4	Profit per ton	5	4			
5						
6						Total profit
7	**Product mix**	3	1,5			21

Fig. 11. **The optimal solution**

The same information may be obtained from the answer report (Fig. 12).

From the "Constraints" table you can determine whether the raw materials are used fully or not. If the raw material has a "binding" status in the appropriate column, it is used fully for the optimal product mix.

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$F$7	Profit per ton ($1000)	0	21

Variable Cells

Cell	Name	Original Value	Final Value
$B$7	Yellow paint	0	3
$C$7	Pink paint	0	1,5

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$4	Raw material, M1	24	$E$4<=$D$4	binding	0
$E$5	Raw material, M2	6	$E$5<=$D$5	binding	0

Fig. 12. **The answer report**

Sensitivity analysis gives you an insight in how the optimal solution changes when you change the coefficients of the model (Fig. 13).

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$7	Yellow paint	3	0	5	1	3
$C$7	Pink paint	1,5	0	4	6	0,666666667

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$2	Raw material, M1	24	0,75	24	12	12
$D$3	Raw material, M2	6	0,5	6	6	2

Fig. 13. **The sensitivity report**

Let's discuss "Variable Cells" table.

If the reduced cost equals 0, the appropriate type of product has a nonzero value. If the reduced cost doesn't equal 0, it is not beneficial to include the appropriate type of product into the optimal product mix.

Suppose that the unit profit of the yellow paint is fixed at its current value of $5. The optimality range for unit profit of the pink paint, that keeps the optimum solution unchanged, may be obtained from the last two columns: [4 – 0.66; 4 + 6].

Suppose that the unit profit of the pink paint is fixed at its current value of $4. The optimality range for unit profit of the yellow paint, that keeps the optimum solution unchanged, may be obtained from the last two columns: [5 – 3; 5 + 1].

Let's discuss the "Constraints" table.

The shadow prices tell us how much the optimal solution can be increased or decreased if we change the right hand side values (resources available) with one unit. This means that a unit increase in the availability of raw material M1 will increase profit by $0.75; a unit increase in the availability of raw material M2 will increase profit by $0.5.

The dual price of $0.75 remains valid for changes in M1 availability within the following boundaries [24 – 12; 24 + 12]. The dual price of $0.5 remains valid for changes in M2 availability within the following boundaries [6 – 2; 6 + 6]. You can take these values from the last two columns.

The final value column in the "Constraints" table shows the used amounts of each raw material. You can see that both resources are used fully.

Tasks for self-study work

A company produces two products. Both products use four raw materials. The maximum daily availability for each material and the raw materials usage rates per unit of product 1 and per unit of product 2 are represented in Table 2. Choose your variant and determine the optimal product mix that maximizes the total daily profit.

Table 2

Variant 1

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	4	1	7
B	1	2	10
C	3	1	6
D	6	1	10
Profit per ton	7	2	

Table 3

Variant 2

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	0	1	4
B	4	1	7
C	2	1	5
D	6	1	10
Profit per ton	4	3	

Table 4

Variant 3

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	1	0	2
B	1	1	6
C	2	1	7
D	4	1	10
Profit per ton	3	2	

Table 5

Variant 4

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	0	1	4
B	4	1	7
C	2	1	5
D	6	1	10
Profit per ton	10	4	

Table 6

Variant 5

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	0	1	5
B	1	0	4
C	2	1	9
D	2	1	6
Profit per ton	2	5	

Table 7

Variant 6

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	3	2	12
B	1	4	20
C	2	1	7
D	1	0	3
Profit per ton	4	1	

Table 8

Variant 7

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	0	1	4
B	2	1	5
C	4	1	7
D	2	0	3
Profit per ton	2	3	

Table 9

Variant 8

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	0	1	4
B	2	1	5
C	4	1	7
D	2	0	3
Profit per ton	6	2	

Table 10

Variant 9

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	0	1	4
B	2	1	5
C	4	1	7
D	2	0	3
Profit per ton	6	1	

Table 11

Variant 10

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	2	3	12
B	1	1	6
C	2	1	8
D	0	1	5
Profit per ton	2	5	

Table 12

Variant 11

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	1	1	6
B	2	1	9
C	I	2	10
D	1	4	8
Profit per ton	2	3	

Table 13

Variant 12

Raw material	Tons of raw material per ton of		Material available
	Product 1	Product 2	
A	1	2	6
B	2	1	7
C	3	1	10
D	0	1	2
Profit per ton	7	3	

Theme 3. Duality and post-optimal analysis

The dual problem is an LP defined directly and systematically from the primal (or original) LP model. The two problems are so closely related that the optimal solution to one problem automatically provides the optimal solution to the other.

How the dual problem is constructed from the primal one:

- a dual variable is defined for each primal constraint;
- a dual constraint is defined for each primal variable;
- the constraint (column) coefficients of a primal variable define the left-hand side coefficients of the dual constraint;
- the objective coefficient of a primal variable define the right-hand side of the dual constraint;
- the objective coefficients of the dual equal the right-hand side of the primal constraint equations.

Example 2. Construct the dual model for the Reddy Mikks problem from Example 1.

Let's define the following objective variables:

y_1 is the cost per unit of the raw material M1;

y_2 is the cost per unit of the raw material M2;

Z is the total cost of the raw materials M1 and M2.

Thus the dual model looks as follows:

$$Z = 24y_1 + 6y_2 \to \min;$$
$$6y_1 + 1y_2 \ge 5;$$
$$4y_1 + 2y_2 \ge 4;$$
$$y_1, y_2 \ge 0.$$

The model we are going to solve looks in Excel as follows (Fig. 14).

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	A	B	C	D	E	F
1		Raw material M1	Raw material M2			Profit per ton
2	Yellow paint	6	1	0	>=	5
3	Pink paint	4	2	0	>=	4
4	Availability	24	6			
5						
6				Total cost		
7	Unit cost	0	0	0		
8						

Fig. 14. **The initial data**

To solve the model, insert the following three SUMPRODUCT functions (Fig. 15).

	A	B	C	D	
1		Raw material M1	Raw material M2		
2	Yellow paint	6	1	=SUMPRODUCT(B2:C2;$B$7:$C$7)	>=
3	Pink paint	4	2	=SUMPRODUCT(B3:C3;$B$7:$C$7)	>=
4	Availability	24	6		
5					
6				Total cost	
7	Unit cost	0	0	=SUMPRODUCT(B4:C4;$B$7:$C$7)	
8					

Fig. 15. **The functions to be inserted**

To find the optimal solution, execute the following steps.

1. On the Data tab, click Solver.

Enter the solver parameters. The result should be consistent with Fig. 16.

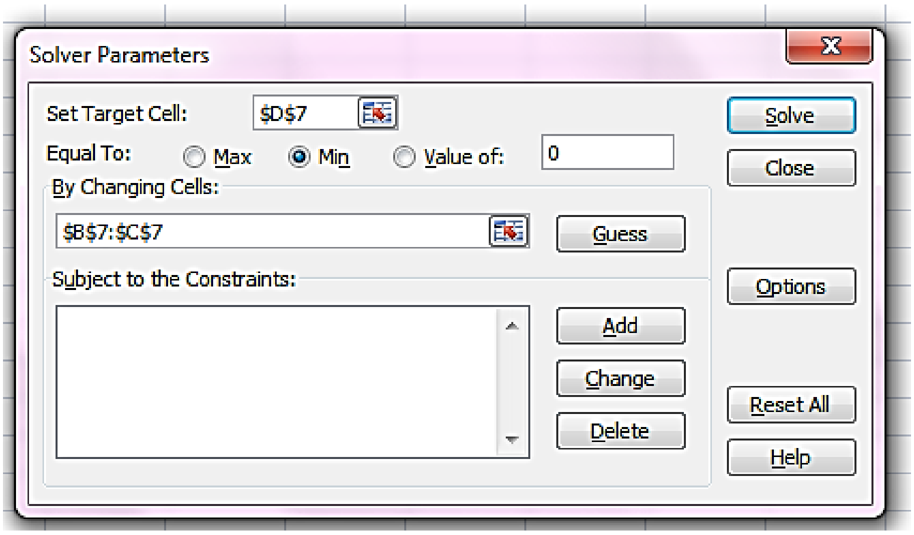


Fig. 16. **The Solver parameters**

2. Enter D7 for the Objective.
3. Click Min.
4. Enter B7:C7 for the Changing Variable Cells.
5. Click Add to enter the constraints (see Fig. 17).

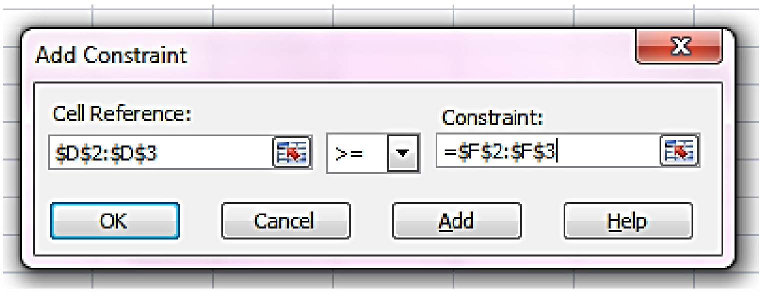


Fig. 17. **The Add constraint**

6. Click Options and select Assume Linear Model and Assume Non-Negative.

7. Finally, click Solve. The result is shown in Fig. 18. Choose Answer and Sensitivity.

	A	B	C	D	E	F
1		Raw material M1	Raw material M2			Profit per ton
2	Yellow paint	6	1	5	>=	5
3	Pink paint	4	2	4	>=	4
4	Availability	24	6			
5						
6				Total cost		
7	Unit cost	0,75	0,5	21		
8						

Fig. 18. **The optimal solution**

The optimal solution of the dual model is:

$$y_1 = 0.75, y_2 = 0.5$$

$$Z = 21$$

The costs of both raw materials needed to produce one unit of the yellow paint (see Cell D2) equal unit profit of the yellow paint (see Cell F2). That is why it is beneficial to include the yellow paint in the optimal product mix.

The costs of both raw materials needed to produce one unit of the pink paint (see Cell D3) equal unit profit of the pink paint (see Cell F3). That is why it is beneficial to include the pink paint in the optimal product mix.

Find coincidences in the reports, obtained for the Primal and the Dual models.

Tasks for self-study work

Choose your variant from the tasks for self-study work for Theme 2.

Construct a dual model.

Solve the dual model with Solver.

Compare the results with those obtained for the tasks for self-study work for Theme 2.

Theme 4. Maths programming applications

The transportation problem

The transportation model is a special class of linear programs that deal with shipping a commodity from sources (e.g., factories) to destinations (e.g., warehouses). The objective is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits.

Example 3. There are three factories that supply cars to three customers. The capacities of the three factories during the next quarter are 100, 200, and 300 cars. The quarterly demand of each customer equals 200 cars. The transportation costs per car on the different routes are given in Table 14.

Table 14

The transportation cost per car

	Customer 1	Customer 2	Customer 3
Factory 1	40	47	80
Factory 2	72	36	58
Factory 3	24	61	71

Use the solver in Excel to find the number of units to ship from each factory to each customer that minimizes the total cost.

Let factories be named sources.

Let customers be named destinations.

Let x_{ij} is the amount shipped from the source *i* to the destination *j*, $i = [1; 3]$, $j = [1; 3]$.

The total transportation cost must be calculated as:

$$40x_{11} + 47x_{12} + 80x_{13} + 72x_{21} + 36x_{22} + 58x_{23} + 24x_{31} + 61x_{32} + 71x_{33}$$

Let's determine the supply restrictions.

The real amount shipped from Factory 1 is $x_{11} + x_{12} + x_{13}$. The amount that must be shipped from Los Angeles is 100. Thus the supply constraint for Los Angeles is

$$x_{11} + x_{12} + x_{13} = 100.$$

The supply constraints for Factory 2 and Factory 3:

$$x_{21} + x_{22} + x_{23} = 200;$$

$$x_{31} + x_{32} + x_{33} = 300.$$

Let's determine the demand restrictions:

The real amount shipped to Customer 1 is $x_{11} + x_{21} + x_{31} = 200$.

The amount that must be shipped to Customer 1 is 200.

Thus the demand constraint for it is

$$x_{11} + x_{21} + x_{31} = 200.$$

The demand constraints for Customer 2 and Customer 3 are

$$x_{12} + x_{22} + x_{32} = 200;$$

$$x_{13} + x_{23} + x_{33} = 200.$$

Variables x_{ij} cannot assume negative values:

$$x_{ij} \geqslant 0, \ i = [1; 3], \ j = [1; 3].$$

The complete model is:

minimize

$$40x_{11} + 47x_{12} + 80x_{13} + 72x_{21} + 36x_{22} + 58x_{23} + 24x_{31} + 61x_{32} + 71x_{33}$$

subject to

$$x_{11} + x_{12} + x_{13} = 100;$$

$$x_{21} + x_{22} + x_{23} = 200;$$

$$x_{31} + x_{32} + x_{33} = 300;$$

$$x_{11} + x_{21} + x_{31} = 200;$$

$$x_{12} + x_{22} + x_{32} = 200;$$

$$x_{13} + x_{23} + x_{33} = 200.$$

$$x_{ij} \geqslant 0, \ i = [1; 3], \ j = [1; 3].$$

These constraints are all equations because the total supply from the three sources (100 + 200 + 300 = 600 cars) equals the total demand at the three destinations (200 + 200 + 200 = 600 cars).

The model we are going to solve looks in Excel as follows (Fig. 19).

	A	B	C	D	E	F	G	H	I	J
1										
2		Unit Cost	Customer 1	Customer 2	Customer 3					
3		Factory 1	40	47	80					
4		Factory 2	72	36	58					
5		Factory 3	24	61	71					
6										
7										
8		Shipments	Customer 1	Customer 2	Customer 3		Total Out		Supply	
9		Factory 1	0	0	0		0	=	100	
10		Factory 2	0	0	0		0	=	200	
11		Factory 3	0	0	0		0	=	300	
12										
13		Total In	0	0	0					
14			=	=	=				Total Cost	
15		Demand	200	200	200				0	
16										

Fig. 19. **The initial data**

Insert the following functions (Fig. 20).

	A	B	C	D	E	F	G	H	I
1									
2		Unit Cost	Customer 1	Customer 2	Customer 3				
3		Factory 1	40	47	80				
4		Factory 2	72	36	58				
5		Factory 3	24	61	71				
6									
7		Shipments	Customer 1	Customer 2	Customer 3		Total Out		Supply
8		Factory 1					=SUM(C8:E8)	=	100
9		Factory 2					=SUM(C9:E9)	=	200
10		Factory 3					=SUM(C10:E10)	=	300
11									
12		Total In	=SUM(C8:C10)	=SUM(D8:D10)	=SUM(E8:E10)				
13			=	=	=				Total Cost
14		Demand	200	200	200				=SUMPRODUCT(C3:E5;C8:E10)
15									

Fig. 20. **The functions to be inserted**

The SUM functions calculate the total shipped from each factory and to each customer. The total cost equals the sum product of the unit cost and shipments.

To find the optimal solution, execute the following steps.

1. On the Data tab, click Solver.

Enter the solver parameters. The result should be consistent with Fig. 21 below.

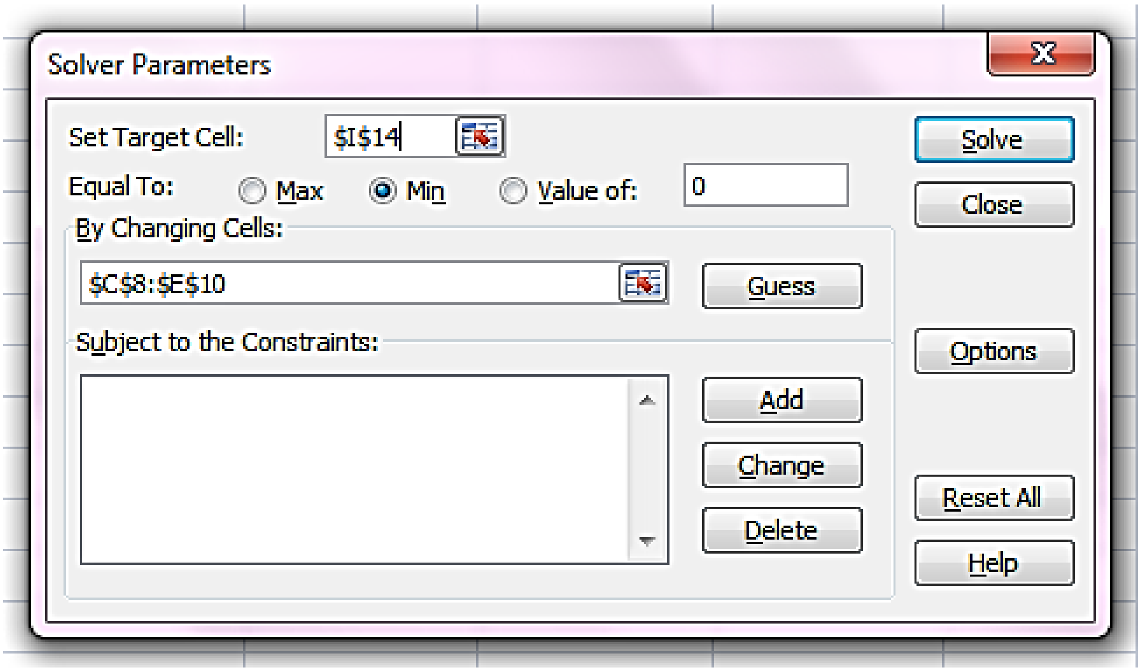


Fig. 21. **The Solver parameters**

2. Enter I14 for the Objective.
3. Click Min.
4. Enter C8:E10 for Changing Variable Cells. Click Add to enter the constraints.
5. Add Supply constraints as shown in Fig. 22.

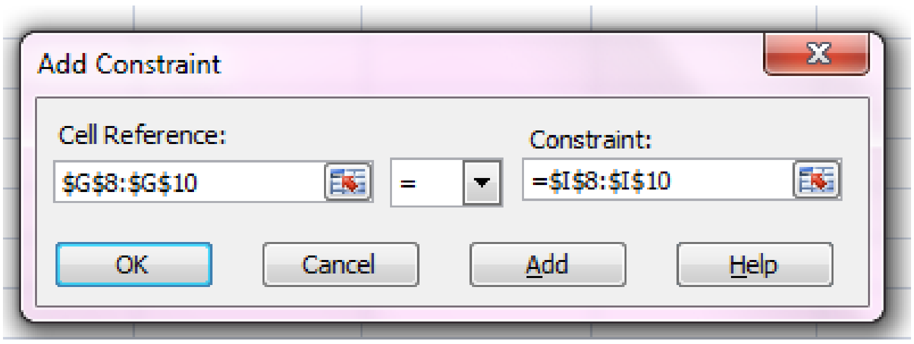


Fig. 22. **The supply constraints**

6. Add demand constraints as shown in Fig. 23.

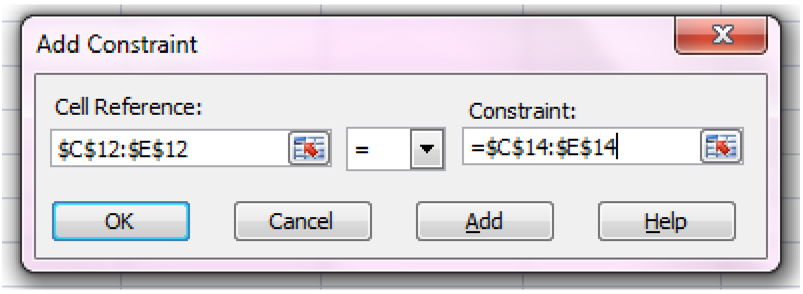


Fig. 23. **The demand constraints**

7. Click Options and select Assume Linear Model and Assume Non-Negative.

8. Finally, click Solve. Choose Answer and Sensitivity reports.

The optimal solution is presented in Fig. 24.

	B	C	D	E	F	G	H	I
1								
2	Unit Cost	Customer 1	Customer 2	Customer 3				
3	Factory 1	40	47	80				
4	Factory 2	72	36	58				
5	Factory 3	24	61	71				
6								
7	Shipments	Customer 1	Customer 2	Customer 3		Total Out		Supply
8	Factory 1	0	100	0		100	=	100
9	Factory 2	0	100	100		200	=	200
10	Factory 3	200	0	100		300	=	300
11								
12	Total In	200	200	200				
13		=	=	=				Total Cost
14	Demand	200	200	200				26000

Fig. 24. **The optimal shipping schedule**

The answer report is presented in Fig. 25.

According to Fig. 25, it is optimal to ship 100 units from Factory 1 to Customer 2; 100 units from Factory 2 to Customer 2; 100 units from Factory 2 to Customer 3; 200 units from Factory 3 to Customer 1, and 100 units from

Factory 3 to Customer 3. This solution gives the minimum cost of 26 000. All constraints are satisfied.

The sensitivity report is presented in Fig. 26.

Suppose that you are to include in the optimal plan the route from Factory 1 to Customer 1. Then you need to reduce the appropriate unit cost. The value of such reduction must be more than 18 (you can see this value in the "Reduced Cost" column).

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$I$14	Demand Total Cost	0	26000

Variable Cells

Cell	Name	Original Value	Final Value
$C$8	Factory 1 Customer 1	0	0
$D$8	Factory 1 Customer 2	0	100
$E$8	Factory 1 Customer 3	0	0
$C$9	Factory 2 Customer 1	0	0
$D$9	Factory 2 Customer 2	0	100
$E$9	Factory 2 Customer 3	0	100
$C$10	Factory 3 Customer 1	0	200
$D$10	Factory 3 Customer 2	0	0
$E$10	Factory 3 Customer 3	0	100

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$G$8	Factory 1 Total Out	100	$G$8=$I$8	binding	0
$G$9	Factory 2 Total Out	200	$G$9=$I$9	binding	0
$G$10	Factory 3 Total Out	300	$G$10=$I$10	binding	0
$C$12	Total In Customer 1	200	$C$12=$C$14	binding	0
$D$12	Total In Customer 2	200	$D$12=$D$14	binding	0
$E$12	Total In Customer 3	200	$E$12=$E$14	binding	0

Fig. 25. **The answer report**

Example 4. Within conditions of example 3 solve the task with the following additional restrictions: Customer 1 must receive at least 50 cars from Factory 1 and at least 10 cars from Factory 2.

The complete model for this example is:

minimize

$$40x_{11} + 47x_{12} + 80x_{13} + 72x_{21} + 36x_{22} + 58x_{23} + 24x_{31} + 61x_{32} + 71x_{33}$$

subject to

$$x_{11} + x_{12} + x_{13} = 100;$$

$$x_{21} + x_{22} + x_{23} = 200;$$

$$x_{31} + x_{32} + x_{33} = 300;$$

$$x_{11} + x_{21} + x_{31} = 200;$$

$$x_{12} + x_{22} + x_{32} = 200;$$

$$x_{13} + x_{23} + x_{33} = 200;$$

$$x_{11} \geq 50;$$

$$x_{21} \geq 10;$$

$$x_{ij} \geqslant 0,\ i = [1; 3],\ j = [1; 3].$$

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$8	Factory 1 Customer 1	0	18	40	1E+30	18
$D$8	Factory 1 Customer 2	100	0	47	11	1E+30
$E$8	Factory 1 Customer 3	0	11	80	1E+30	11
$C$9	Factory 2 Customer 1	0	61	72	1E+30	61
$D$9	Factory 2 Customer 2	100	0	36	12	11
$E$9	Factory 2 Customer 3	100	0	58	11	12
$C$10	Factory 3 Customer 1	200	0	24	18	1E+30
$D$10	Factory 3 Customer 2	0	12	61	1E+30	12
$E$10	Factory 3 Customer 3	100	0	71	12	18

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$G$8	Factory 1 Total Out	100	47	100	0	100
$G$9	Factory 2 Total Out	200	36	200	0	100
$G$10	Factory 3 Total Out	300	49	300	0	100
$C$12	Total In Customer 1	200	-25	200	100	0
$D$12	Total In Customer 2	200	0	200	0	1E+30
$E$12	Total In Customer 3	200	22	200	100	0

Fig. 26. **The sensitivity report**

Add right-hand side values for additional restrictions as shown in Fig. 27.

	J	K	L	M	N
6			Additional restrictions		
7		Shipments	Customer 1	Customer 2	Customer 3
8		Factory 1	50	0	0
9		Factory 2	10	0	0
10		Factory 3	0	0	0
11					

Fig. 27. **The additional restrictions**

The answer is shown in Fig. 28.

Conclusion: it is optimal to ship 50 units from Factory 1 to Customer 1; 50 units from Factory 1 to Customer 2; 10 units from Factory 2 to Customer 1; 150 units from Factory 2 to Customer 2; 40 units from Factory 2 to Customer 3; 140 units from Factory 3 to Customer 1, and 160 units from Factory 3 to Customer 3.

As you can see in Fig. 28, the results are worse than the previous ones for the initial model without additional restrictions. The total cost of shipments equals 27 510. This is more than the total cost for the initial model (26 000).

N18

	A	B	C	D	E	F	G	H	I	J
6										
7		Shipments	Customer 1	Customer 2	Customer 3		Total Out		Supply	
8		Factory 1	50	50	0		100	=	100	
9		Factory 2	10	150	40		200	=	200	
10		Factory 3	140	0	160		300	=	300	
11										
12		Total In	200	200	200					
13			=	=	=				Total Cost	
14		Demand	200	200	200				27510	
15										

Fig. 28. **The optimal shipping schedule for the additional restrictions**

Balancing the Transportation Model

The transportation algorithm is based on the assumption that the model is balanced, meaning that the total demand equals the total supply. If the model is unbalanced, we can always add a dummy source or a dummy destination to restore the balance:

• If the total supply exceeds the total demand, we can balance the transportation problem by creating a dummy demand point that has a demand equal to the amount of the excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate the unused supply capacity.

• If the total supply is less than the total demand, actually the problem has no feasible solution. To solve the problem it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, a penalty is often associated with the unmet demand. This means that a dummy supply point should be introduced.

Example 5. JRU Auto has three plants in Los Angeles, Detroit, and New Orleans, and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are 1000, 1500, and 500 cars. The quarterly demands at the two distribution centers are 2300 and 1400 cars.

The transportation costs per car on the different routes are given in the Table 14.

Table 14

The transportation costs

Unit Cost	Denver (1)	Miami (2)
Los Angeles (1)	80	215
Detroit (2)	100	108
New Orleans (3)	102	68

The objective is to determine the shipping schedule that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

The solution

The model is unbalanced. The sum of capacities is more than the sum of demands. That is why a dummy source must be included into the model.

The complete model for this example is

$$Z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32} \to \min;$$

$$x_{11} + x_{12} = 1000;$$

$$x_{21} + x_{22} = 1500;$$

$$x_{31} + x_{32} = 500;$$

$$x_{41} + x_{42} = 700;$$

$$x_{11} + x_{21} + x_{31} + x_{41} = 2300;$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 1400;$$

$$x_{ij} \geqslant 0,\ i = [1;\ 4],\ j = [1;\ 2].$$

The model we are going to solve looks as follows in Excel (Fig. 29).

	A	B	C	D	E	F	G	H	I	J	K
2		Unit Cost	Denver (1)	Miami (2)							
3		Los Angeles (1)	80	215							
4		Detroit (2)	100	108							
5		New Orleans (3)	102	68							
6		Dummy source	673	673							
7											
8		Shipments	Denver (1)	Miami (2)		Total Out		Supply			
9		Los Angeles (1)	0	0		0	=	1000			
10		Detroit (2)	0	0		0	=	1500			
11		New Orleans (3)	0	0		0	=	500			
12		Dummy source	0	0		0	=	700		sum of demands	
13										3000	
14		Total In	0	0							
15			=	=				Total Cost			
16		Demand	2300	1400				0			
17											
18		sum of capacities		3700							
19											

Fig. 29. **The initial data**

Since shipments to the dummy destination (or from the dummy source) are not real shipments, they are assigned a very large cost. This makes them not advantageous (Fig. 30).

	A	B	C	D
1				
2		**Unit Cost**	**Denver (1)**	**Miami (2)**
3		**Los Angeles (1)**	**80**	**215**
4		**Detroit (2)**	**100**	**108**
5		**New Orleans (3)**	**102**	**68**
6		**Dummy source**	**=SUM($C$3:$D$5)**	**=SUM($C$3:$D$5)**
7				
8		**Shipments**	**Denver (1)**	**Miami (2)**
9		**Los Angeles (1)**	**1000**	**0**
10		**Detroit (2)**	**1300**	**200**
11		**New Orleans (3)**	**0**	**500**
12		**Dummy source**	**0**	**700**
13				
14		**Total In**	**=SUM(C9:C12)**	**=SUM(D9:D12)**
15			**=**	**=**
16		**Demand**	**2300**	**1400**
17				
18		**sum of capacities**		**=SUM(C16:D16)**

Fig. 30. **Determining the unit costs for the dummy source**

Determine the supply value for the dummy source as shown in Fig. 31.

	F	G	H	I	J
7					
8	**Total Out**		**Supply**		
9	**=SUM(C9:D9)**	**=**	**1000**		
10	**=SUM(C10:D10)**	**=**	**1500**		
11	**=SUM(C11:D11)**	**=**	**500**		
12	**=SUM(C12:D12)**	**=**	**=D18-J13**		**sum of demands**
13					**=SUM(H9:H11)**
14					
15			**Total Cost**		
16			**=SUMPRODUCT(C3:D5;C9:D11)**		

Fig. 31. **The supply value for the dummy source**

The optimal solution is shown in Fig. 32.

	A	B	C	D	E	F	G	H	I	J	K
7											
8		Shipments	Denver (1)	Miami (2)		Total Out		Supply			
9		Los Angeles (1)	1000	0		1000	=	1000			
10		Detroit (2)	1300	200		1500	=	1500			
11		New Orleans (3)	0	500		500	=	500			
12		Dummy source	0	700		700	=	700		sum of demands	
13										3000	
14		Total In	2300	1400							
15			=	=				Total Cost			
16		Demand	2300	1400				265600			
17											
18		sum of capacities		3700							

Fig. 32. **The optimal solution**

According to Fig. 32, it is optimal to ship 1000 cars from Los Angeles to Denver, 1300 cars from Detroit to Denver, 200 cars from Detroit to Miami; 500 cars from New Orleans to Miami. 700 cars are shipped from the Dummy source to Miami. It means that Miami demand will not be satisfied by 700 cars. The optimum value of the total cost equals 265 600.

Example 6. Within the conditions of the previous example let the capacities of the three plants during the next quarter be 1000, 1500, and 500 cars. Let the quarterly demands at the two distribution centers be 1000 and 1400 cars. The transportation costs per car on the different routes are given in Table 14.

The objective is to determine the shipping schedule that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

The model is unbalanced. The sum of capacities is less than the sum of demands. That is why a dummy destination must be included into the model.

The complete model for this example is

$$Z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32} \to \min;$$

$$x_{11} + x_{12} = 1000;$$

$$x_{21} + x_{22} = 1500;$$

$$x_{31} + x_{32} = 500;$$

$$x_{41} + x_{42} = 700;$$

$$x_{11} + x_{21} + x_{31} + x_{41} = 2300;$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 1400;$$

$$x_{ij} \geqslant 0,\ i = [1;\ 4],\ j = [1;\ 2].$$

The model we are going to solve looks as follows in Excel (Fig. 33).

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		Unit Cost	Denver (1)	Miami (2)	Dummy destination							
3		Los Angeles (1)	80	215	673							
4		Detroit (2)	100	108	673							
5		New Orleans (3)	102	68	673							
6												
7		Shipments	Denver (1)	Miami (2)	Dummy destination		Total Out		Supply			
8		Los Angeles (1)	0	0	0		0	=	1000			
9		Detroit (2)	0	0	0		0	=	1500			
10		New Orleans (3)	0	0	0		0	=	500		sum of demands	
11											3000	
12		Total In	0	0	0							
13			=	=	=				Total Cost			
14		Demand	1000	1400	600				0			
15												
16			sum of capacities		2400							
17												

Fig. 33. **The initial data**

	A	B	C	D	E	F
1						
2		**Unit Cost**	**Denver (1)**	**Miami (2)**	**Dummy destination**	
3		**Los Angeles (1)**	**80**	**215**	**=SUM($C$3:$D$5)**	
4		**Detroit (2)**	**100**	**108**	**=SUM($C$3:$D$5)**	
5		**New Orleans (3)**	**102**	**68**	**=SUM($C$3:$D$5)**	
6						
7		**Shipments**	**Denver (1)**	**Miami (2)**	**Dummy destination**	
8		**Los Angeles (1)**	**0**	**0**	**0**	
9		**Detroit (2)**	**0**	**0**	**0**	
10		**New Orleans (3)**	**0**	**0**	**0**	
11						
12		**Total In**	**=SUM(C8:C10)**	**=SUM(D8:D10)**	**=SUM(E8:E10)**	
13			**=**	**=**	**=**	
14		**Demand**	**1000**	**1400**	**=K11-E16**	
15						
16			**sum of capacities**		**=SUM(C14:D14)**	

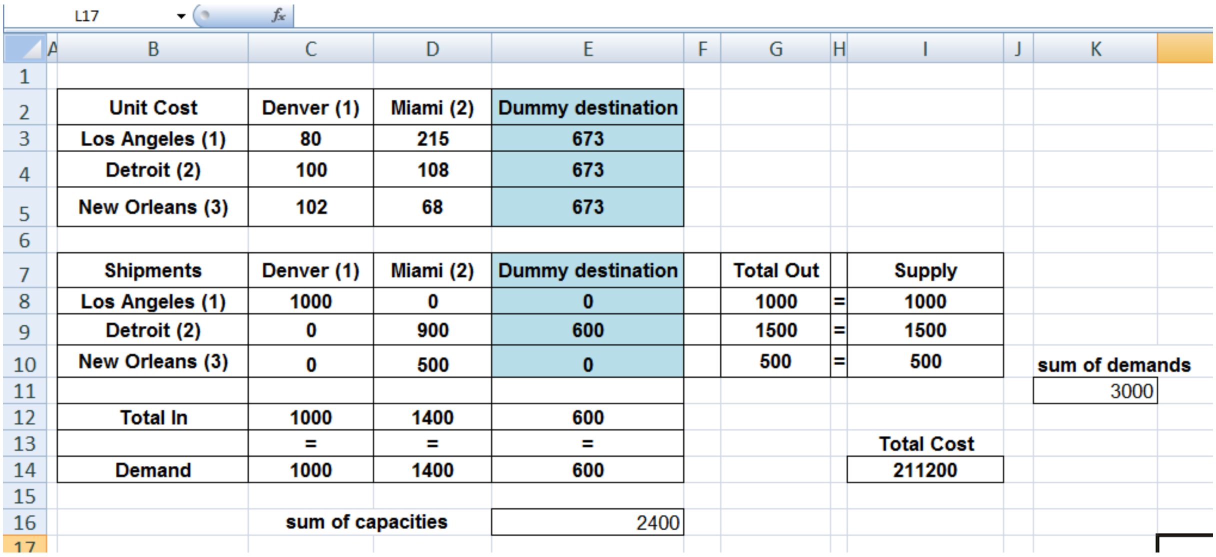
Fig. 34. **Determining the unit costs for the dummy destination**

	F	G	H	I	J	K
6						
7		**Total Out**		**Supply**		
8		**=SUM(C8:E8)**	**=**	**1000**		
9		**=SUM(C9:E9)**	**=**	**1500**		
10		**=SUM(C10:E10)**	**=**	**500**		**sum of demands**
11						**=SUM(I8:I10)**
12						
13				**Total Cost**		
14				**=SUMPRODUCT(C3:D5;C8:D10)**		

Fig. 35. **The value for the dummy destination**

The optimal solution is shown in Fig. 36.

According to Fig. 36, it is optimal to ship 1000 cars from Los Angeles to Denver; 900 cars from Detroit to Miami; 500 cars from New Orleans to Miami. 600 cars are shipped from Detroit to the dummy destination. It means that capacities of Detroit will not be used fully. The unused amount equals 600 cars. The optimum value of the total cost equals 211 200.



Unit Cost	Denver (1)	Miami (2)	Dummy destination
Los Angeles (1)	80	215	673
Detroit (2)	100	108	673
New Orleans (3)	102	68	673

Shipments	Denver (1)	Miami (2)	Dummy destination	Total Out		Supply
Los Angeles (1)	1000	0	0	1000	=	1000
Detroit (2)	0	900	600	1500	=	1500
New Orleans (3)	0	500	0	500	=	500
Total In	1000	1400	600			
	=	=	=			
Demand	1000	1400	600			

sum of demands 3000

Total Cost 211200

sum of capacities 2400

Fig. 36. **The optimal solution**

Tasks for self-study work

There are three factories that supply cars to four customers. The transportation costs per car on the different routes are given in the appropriate table below.

Choose your variant.

Form a math model and determine the shipping schedule that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

Variant 1

Table 15

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	1	6	8	5	20
Factory 2	12	2	3	11	40
Factory 3	4	7	9	10	70
Demand	50	30	25	15	

Solve the first model with additional restrictions: Customer 2 must receive precisely 4 cars from Factory 2 and Factory 3 must ship at least 6 cars to Customer 4. Compare the results.

Variant 2

Table 16

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	12	6	9	5	15
Factory 2	1	11	3	2	45
Factory 3	4	7	8	10	80
Demand	50	30	25	15	

Solve the first model with additional restrictions: Customer 3 must receive precisely 10 cars from Factory 2 and Factory 3 must ship at most 26 cars to Customer 3. Compare the results.

Variant 3

Table 17

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	1	6	8	5	20
Factory 2	12	2	3	11	40
Factory 3	4	7	9	10	50
Demand	50	30	25	15	

Solve the first model with additional restrictions: Customer 2 must receive at most 40 cars from Factory 1 and Factory 3 must ship precisely 20 cars to Customer 4. Compare the results.

Variant 4

Table 18

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	1	6	8	5	35
Factory 2	12	2	3	11	25
Factory 3	4	7	9	10	45
Demand	40	40	30	15	

Solve the first model with additional restrictions: Customer 3 must receive at least 5 cars from Factory 2 and Factory 1 must ship at most 20 cars to Customer 1. Compare the results.

Variant 5

Table 19

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	2	7	9	6	40
Factory 2	13	3	4	12	30
Factory 3	5	8	10	11	50
Demand	50	50	40	25	

Solve the first model with additional restrictions: Customer 2 must receive at least 25 cars from Factory 1 and Factory 1 must ship precisely 20 cars to Customer 4. Compare the results.

Variant 6

Table 20

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	2	7	9	6	25
Factory 2	13	3	4	12	45
Factory 3	5	8	10	11	75
Demand	45	25	20	10	

Solve the first model with additional restrictions: Customer 2 must receive at most 25 cars from Factory 2 and Factory 3 must ship at least 20 cars to Customer 4. Compare the results.

Variant 7

Table 21

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	14	8	11	7	15
Factory 2	3	13	5	4	45
Factory 3	6	9	10	12	80
Demand	50	30	25	15	

Solve the first model with additional restrictions: Customer 3 must receive precisely 5 cars from Factory 2 and Factory 1 must ship at least 8 cars to Customer 1. Compare the results.

Variant 8

Table 22

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	2	7	9	6	20
Factory 2	13	3	4	12	40
Factory 3	5	8	10	11	50
Demand	55	35	30	20	

Solve the first model with additional restrictions: Customer 4 must receive precisely 8 cars from Factory 1 and Factory 2 must ship at most 23 cars to Customer 1. Compare the results.

Variant 9

Table 23

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	4	9	11	8	35
Factory 2	15	5	6	14	25
Factory 3	7	10	12	13	45
Demand	35	35	25	10	

Solve the first model with additional restrictions: Customer 2 must receive precisely 12 cars from Factory 2 and Factory 1 must ship at least 20 cars to Customer 1. Compare the results.

Variant 10

Table 24

The initial data

Sources	Destinations				Supply
	Customer 1	Customer 2	Customer 3	Customer 4	
Factory 1	5	10	12	9	35
Factory 2	16	6	7	15	25
Factory 3	8	11	13	14	45
Demand	45	45	35	20	

Solve the first model with additional restrictions: Customer 4 must receive at most 10 cars from Factory 2 and Factory 3 must ship at least 20 cars to Customer 1. Compare the results.

The Assignment Problem

Example 7. Shop2Shop needs to assign 3 tasks to *s* persons. The cost of performing a task is a function of the skills of the persons. Table 25 summarizes the cost of the assignments. Determine the optimal assignment, that minimizes the total costs.

Table 25

Assignment costs

Cost	Task 1	Task 2	Task 3
Person 1	43	38	82
Person 2	75	27	60
Person 3	27	52	73

Use the Solver to find the assignment of persons to tasks that minimizes the total cost.

Let's formulate the math model:

x_{ij} is an assignment of Person *i* to Task *j*, $i, j = [1; 4]$

$$x_{ij} = \begin{cases} 1, & \text{if Person } i \text{ is assigned to Task } j \\ 0, & \text{if Person } i \text{ isn't assigned to Task } j \end{cases}$$

Z is the total cost of assignment.

$$Z = 43x_{11} + 38x_{12} + 82x_{13} + 75x_{21} + 27x_{22} + 60x_{23} + 27x_{31} + 52x_{32} + 73x_{33} \to \min.$$

Constraints for the Persons:

$$x_{11} + x_{12} + x_{13} = 1;$$

$$x_{21} + x_{22} + x_{23} = 1;$$

$$x_{31} + x_{32} + x_{33} = 1.$$

Constraints for the Tasks:

$$x_{11} + x_{21} + x_{31} = 1;$$

$$x_{12} + x_{22} + x_{32} = 1;$$

$$x_{13} + x_{23} + x_{33} = 1.$$

The model we are going to solve looks as follows in Excel (Fig. 37).

	A	B	C	D	E	F	G	H	I	J
1										
2		Cost	Task 1	Task 2	Task 3					
3		Person 1	43	38	82					
4		Person 2	75	27	60					
5		Person 3	27	52	73					
6										
7										
8		Assignment	Task 1	Task 2	Task 3		Tasks Assigned		Supply	
9		Person 1					0	=	1	
10		Person 2					0	=	1	
11		Person 3					0	=	1	
12										
13		Persons Assigned	0	0	0					
14			=	=	=				Total Cost	
15		Demand	1	1	1				0	
16										
17										

Fig. 37. **The initial data**

To cope with the task, name the ranges as shown in Table 26.

Table 26

The names of the cell ranges

Range name	Cells
Cost	C3:E5
Assignment	C9:E11
Persons assigned	C13:E13
Demand	C15:E15
Tasks assigned	G9:G11
Supply	I9:I11
Total cost	I15

Enter SUM and SUMPRODUCT functions as shown in Fig. 38.

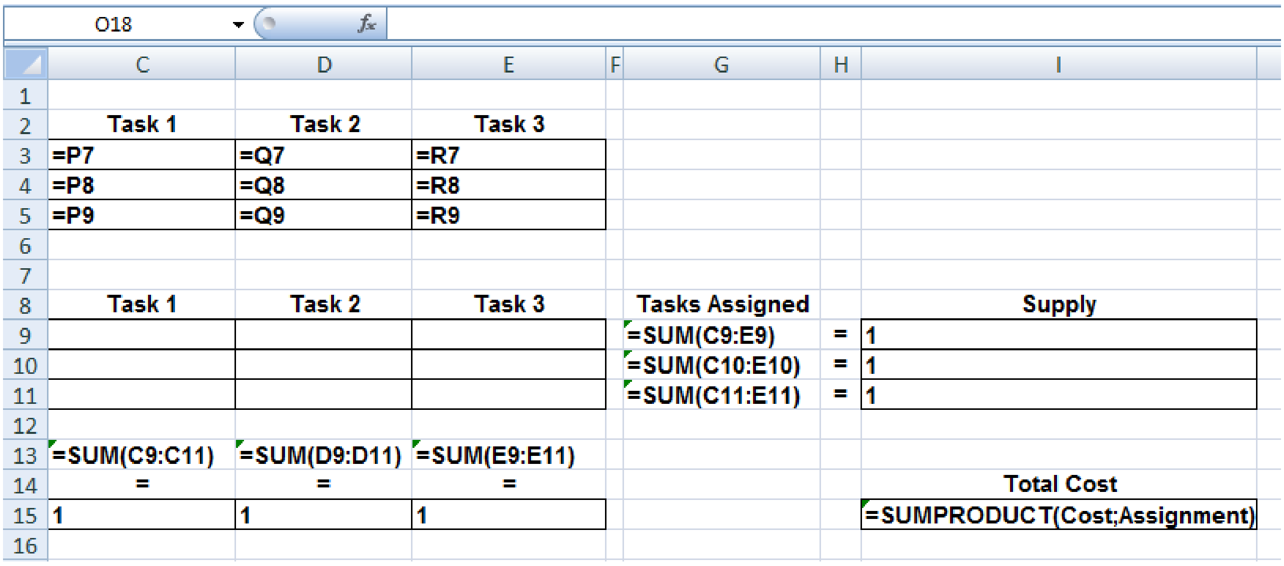


Fig. 38. **The functions**

1. Enter the Solver parameters. The result should be consistent with Fig. 39.

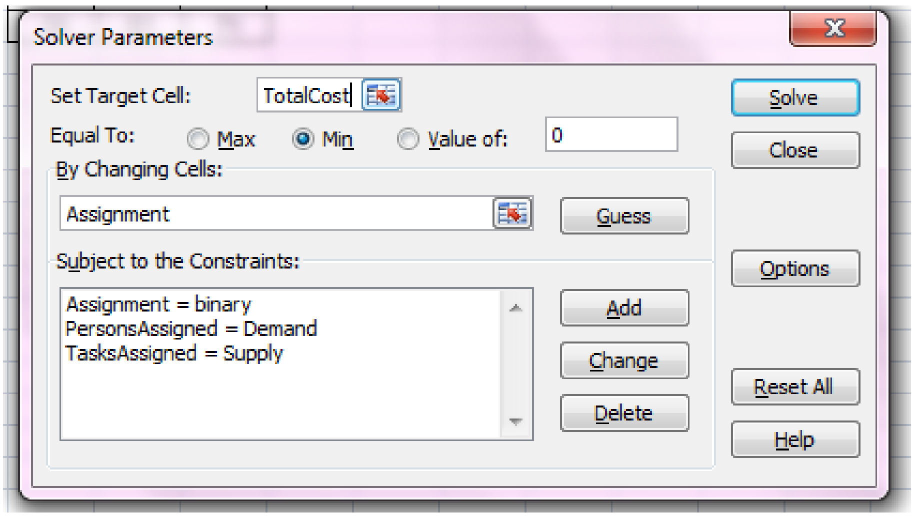


Fig. 39. **The Solver parameters**

2. Enter Total Cost for the Objective.
3. Click Min.
4. Enter Assignment for the Changing Variable Cells.
5. Click Add to enter the following constraint (Fig. 40).

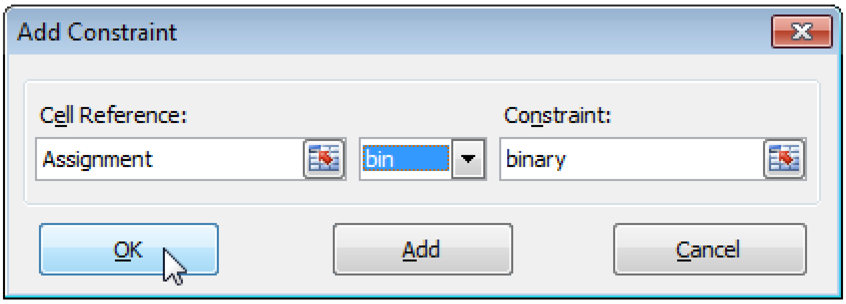


Fig. 40. **The Binary constraint**

Note: binary variables are either 0 or 1.
6. Click Add to enter the following constraint (Fig. 41).

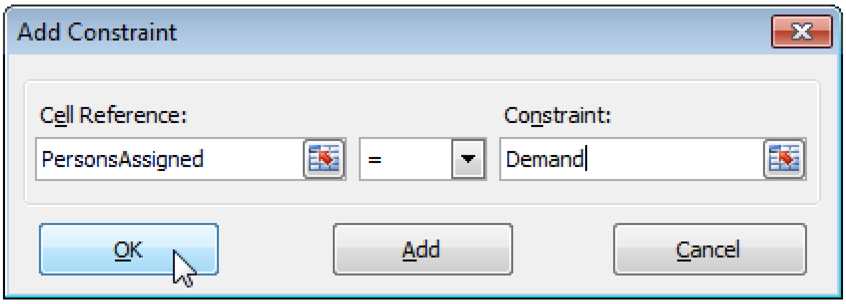


Fig. 41. **The demand constraints**

7. Click Add to enter the following constraint (Fig. 42).

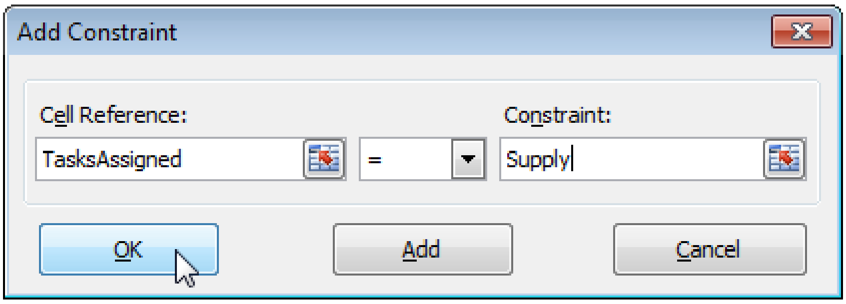
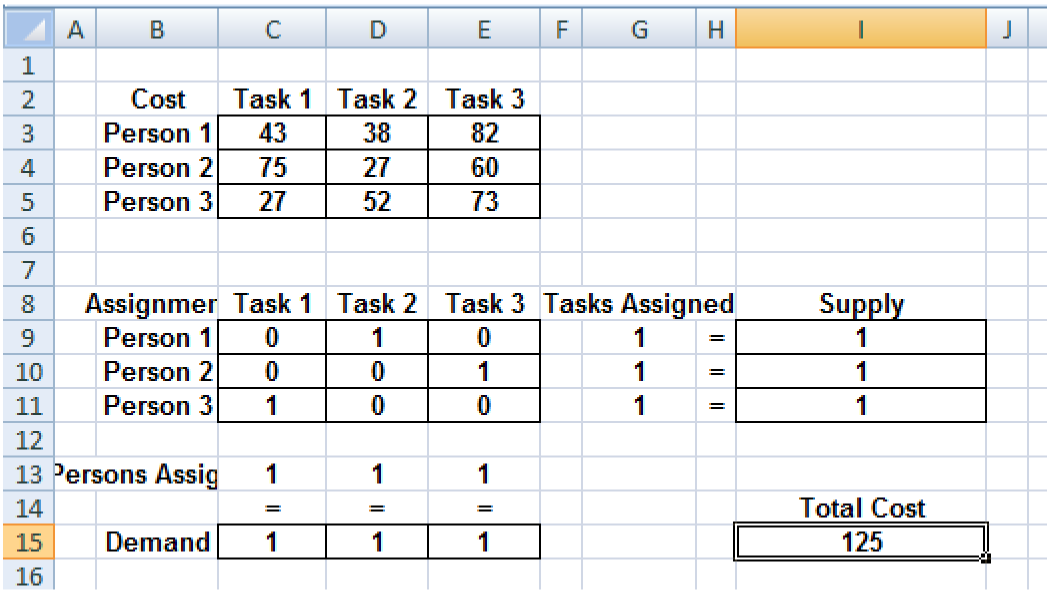


Fig. 42. **The supply constraints**

8. Click Options and select Assume Linear Model and Assume Non-Negative.

9. Finally, click Solve.

The optimal solution is presented in Fig. 43.



	A	B	C	D	E	F	G	H	I	J
1										
2		Cost	Task 1	Task 2	Task 3					
3		Person 1	43	38	82					
4		Person 2	75	27	60					
5		Person 3	27	52	73					
6										
7										
8		Assignmer	Task 1	Task 2	Task 3	Tasks Assigned			Supply	
9		Person 1	0	1	0		1	=	1	
10		Person 2	0	0	1		1	=	1	
11		Person 3	1	0	0		1	=	1	
12										
13	Persons Assig		1	1	1					
14			=	=	=				Total Cost	
15		Demand	1	1	1				125	
16										

Fig. 43. **The optimal assignment**

According to Fig. 43, it is optimal to assign Person 1 to task 2; Person 2 to Task 3, and Person 3 to Task 1. This solution gives the minimum cost of 125. All constraints are satisfied.

Tasks for self-study work

You are to assign 5 tasks to 5 persons. The cost of performing a task is the function of the skills of the persons. Tables 27 – 36 summarize the cost of the assignments.

Choose you variant.

Determine the optimal assignment that minimizes the total costs.

Variant 1

Table 27

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	40	24	80	67	42
Person 2	19	23	76	35	32
Person 3	57	16	55	36	61
Person 4	24	55	71	40	55
Person 5	60	24	54	46	45

Variant 2

Table 28

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	34	42	80	65	35
Person 2	16	32	76	53	36
Person 3	24	61	55	45	40
Person 4	54	55	71	54	46
Person 5	60	55	54	65	35

Variant 3

Table 29

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	65	47	34	42	42
Person 2	53	42	16	32	32
Person 3	45	32	24	61	61
Person 4	54	61	54	55	55
Person 5	65	55	54	46	45

Variant 4

Table 30

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	40	24	80	67	42
Person 2	19	23	76	35	32
Person 3	57	16	55	36	61
Person 4	24	55	71	40	55
Person 5	60	24	54	46	45

Variant 5

Table 31

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	34	42	80	65	35
Person 2	16	32	76	53	36
Person 3	24	61	55	45	40
Person 4	54	55	71	54	46
Person 5	60	55	54	65	35

Variant 6

Table 32

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	65	47	34	42	42
Person 2	53	42	16	32	32
Person 3	45	32	24	61	61
Person 4	54	61	54	55	55
Person 5	65	55	54	46	45

Variant 7

Table 33

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	25	74	34	42	42
Person 2	35	24	38	20	19
Person 3	54	18	24	61	37
Person 4	28	61	19	49	55
Person 5	65	55	54	46	45

Variant 8

Table 34

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	49	28	80	67	42
Person 2	19	34	67	35	17
Person 3	75	65	55	43	61
Person 4	24	50	17	40	87
Person 5	62	24	45	55	45

Variant 9

Table 35

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	28	74	42	67	45
Person 2	34	24	32	35	18
Person 3	65	18	61	36	16
Person 4	50	61	55	40	34
Person 5	24	55	45	46	45

Variant 10

Table 36

The cost of performing the task

Cost	Task 1	Task 2	Task 3	Task 4	Task 5
Person 1	38	42	16	65	38
Person 2	16	19	67	53	36
Person 3	24	37	55	54	78
Person 4	54	55	71	57	46
Person 5	60	45	54	65	53

Theme 5. Markov chains

Let x_1 be a random variable that characterizes the state of the system at discrete points in time *t* = 1, 2, ...

The number of states in a stochastic process may be finite or infinite.

The family of random variables $\{x_t\}$forms a stochastic process.

A stochastic process is a Markov process if the occurrence of a future state depends only on the immediately preceding state. This means that given the chronological times $t_0, t_1, ..., t_n$ the family of random variables $\{X_{t_n}\} = \{x_1, ..., x_n\}$ is said to be a Markov process if it possesses the following property: $P(X_{t_n} = x_n \mid = X_{t_{n-1}} = x_{n\ 1}, ..., X_{t_0} = x_0) = P(X_{t_n} = x_n \mid = X_{t_{n-1}} = x_{n-1},)$.

In a Markovian process with *n* exhaustive and mutually exclusive states (outcomes) $P_i(t)$ is defined as the **absolute probability** of the *i*-th state at the step *t*.

In a Markovian process with *n* exhaustive and mutually exclusive states (outcomes), the probabilities at a specific point in time *t* = 0, 1, 2, ... is usually written as $p_{ij} = P(X_t = j \mid = X_{t-1} = i),\ i, j = 1, 2, ..., n,\ t = 0, 1, 2, ..., T.$

This is known as the one-step transition probability of moving from state *i* at *t* - 1 to state *j* at *t.*

By definition, we have

$$\sum_j p_{ij} = 1,\ i = 1, 2, ..., n.$$

$$p_{ij} \geq 0,\ i, j = 1, 2, ..., n$$

A convenient way for summarizing the one-step transition probabilities is to use the following matrix notation:

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nj} & \cdots & p_{nn} \end{pmatrix}$$

The matrix *P* defines the so-called Markov chain. It has the property that all its transition probabilities p_{ij} are fixed (stationary) and independent over time.

Given the vector of initial absolute probabilities $P(0) = (P_1(0); P_2(0); ...; P_n(0))$ of starting in states 1, 2, …, *n* and the transition matrix *P* of a Markov chain, the vector of absolute probabilities $P(k) = (P_1(k); P_2(k); ...; P_n(k))$ of being in each state after k steps *(k* > 0) are computed as follows:

$$P(k) = P(0)P^k$$

or

$$P(k) = P(k-1)P.$$

Example 8. The Land of Oz is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow, the next day may be rainy with probability 0.1 or nice with probability 0.4. If they have rain, the next day may be snowy with probability 0.2 or rainy with probability 0.5. Represent the situation as a Markov chain.

Today is Wednesday and the weather is rainy. Determine the absolute probabilities of the three states of weather on Friday.

With this information we form a Markov chain as follows. We take as states the kinds of weather *R, N,* and *S*. From the above information we determine the transition probabilities:

$$\begin{array}{c} \\ P = \end{array} \begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0.4 & 0.5 \end{pmatrix} \end{array} \begin{array}{c} \\ R \\ N \\ S \end{array}$$

The entries in the first row of the matrix *P* epresent the probabilities for the various kinds of weather following a rainy day. Similarly, the entries in the second and third rows represent the probabilities for the various kinds of weather following nice and snowy days, respectively.

The initial condition of the weather is rainy. Let's form the vector of initial probabilities:

$$k = 0 \qquad P(0) = (P_R(0);\ P_N(0);\ P_S(0)) = (1; 0; 0)$$

The model we are going to solve looks as follows in Excel (Fig. 43).

	A	B	C	D	E	F
1						
2			one-step transition probabilities			
3		rainy	0,50	0,30	0,20	
4		nice	0,50	0,00	0,50	
5		snowy	0,10	0,40	0,50	
6			rainy	nice	snowy	
7						
8			vector of initial probabilities			
9			1	0	0	
10			rainy	nice	snowy	
11						

Fig. 43. **The initial data**

To cope with the task it is crucial to know how to work with matrix operations in MS Excel.

The key to understanding the use of matrix operations in Excel is the concept of the *Matrix (Array) formula*. Such a formula uses matrix operations and returns a result that can be a matrix, a vector, or a scalar, depending on the computations involved. Whatever the result may be, an area on the spreadsheet **of precisely the correct size** must be selected before the formula is typed in (otherwise you will either lose some of the answers or get added and possibly confusing information).

After typing such a formula, you "enter" it with three keys pressed at once: CTRL, SHIFT and ENTER. This indicates that a matrix (array) result is really desired. It also designates the entire selected range as the desired location for the answer. To modify or delete the formula, select the entire region beforehand.

Let's calculate the vector of probabilities on Thursday:

1. For the **range** of cells C12:E12 click the *functions* button and choose MMULT. Fill the appropriate ranges according to Fig. 44.
2. **Click at once**: CTRL, SHIFT and ENTER.

	B	C	D	E
7				
8		vector of initial probabilities		
9		1	0	0
10		rainy	nice	snowy
11		vector of probabilities on Thursday		
12		=MMULT(C9:E9;C3:E5)		
13				

Fig. 44. **The probabilities on Thursday**

Let's calculate the vector of probabilities on Friday.

1. For the **range** of cells C15:E15 click the *functions* button and choose MMULT. Fill the appropriate ranges according to Fig. 45.
2. Click **at once**: CTRL, SHIFT and ENTER.

13				
14		vector of probabilities on Friday		
15		=MMULT(C12:E12;C3:E5)		
16				

Fig. 45. **The probabilities on Friday**

The results are shown in Fig. 46.

	B	C	D	E	F
1					
2		**one-step transition probabilities**			
3	rainy	0,50	0,30	0,20	
4	nice	0,50	0,00	0,50	
5	snowy	0,10	0,40	0,50	
6		rainy	nice	snowy	
7					
8		vector of initial probabilities			
9		1	0	0	
10		rainy	nice	snowy	
11		vector of probabilities on Thursday			
12		0,5	0,3	0,2	
13					
14		vector of probabilities on Friday			
15		0,42	0,23	0,35	

Fig. 46. **The final solution**

Conclusion: the probability of rain is 0.42; the probability of nice weather is 0.23; the probability of snow is 0.35.

Another way to determine the state of weather is to calculate two-steps transition probabilities:

1. For the **range** of cells I3:K5 choose MMULT (Fig. 47).

	H	I	J	K
2		two-steps transition probabilities		
3		=MMULT(C3:E5;C3:E5)		
4				
5				

Fig. 47. **Two-step transition probabilities**

2. For the **range** of cells I15:K15 choose MMULT (Fig. 48).

	H	I	J	K
12				
13				
14		vector of probabilities on Friday		
15		=MMULT(C9:E9;I3:K5)		

Fig. 48. **The probabilities on Friday**

The results are shown in Fig. 49.

	B	C	D	E	F	G	H	I	J	K	L
2		one-step transition probabilities						two-steps transition probabilities			
3	rainy	0,50	0,30	0,20				0,42	0,23	0,35	
4	nice	0,50	0,00	0,50				0,3	0,35	0,35	
5	snowy	0,10	0,40	0,50				0,3	0,23	0,47	
6		rainy	nice	snowy							
7											
8		vector of initial probabilities									
9		1	0	0							
10		rainy	nice	snowy							
11		vector of probabilities on Thursday									
12		0,5	0,3	0,2							
13											
14		vector of probabilities on Friday						vector of probabilities on Friday			
15		0,42	0,23	0,35				0,42	0,23	0,35	
16											

Fig. 49. **Two ways of calculating absolute probabilities**

Tasks for self-study work

A company may occur in some of the following three states: normal, pre-crisis, and crisis. The transition matrix is given below.

Find the absolute probabilities of each state in May.

Variant 1

Table 37

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.11	0.49	0.4
Pre-crisis	0.08	0.35	0.57
Crisis	0.07	0.12	0.81

In March the company was in the normal state.

Variant 2

Table 38

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.21	0.39	0.4
Pre-crisis	0.18	0.25	0.22
Crisis	0.17	0.02	0.41

In March the company was in the pre-crisis state.

Variant 3

Table 39

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.31	0.39	0.3
Pre-crisis	0.28	0.25	0.47
Crisis	0.27	0.02	0.71

In March the company was in the crisis state.

Variant 4

Table 40

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.4	0.49	0.11
Pre-crisis	0.22	0.35	0.43
Crisis	0.41	0.12	0.47

In March the company was in the normal state.

Variant 5

Table 41

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.54	0.22	0.24
Pre-crisis	0.81	0.05	0.14
Crisis	0.34	0.12	0.54

In March the company was in the pre-crisis state.

Variant 6

Table 42

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.4	0.3	0.3
Pre-crisis	0.22	0.05	0.73
Crisis	0.09	0.08	0.83

In March the company was in the crisis state.

Variant 7

Table 43

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.39	0.22	0.39
Pre-crisis	0.25	0.05	0.7
Crisis	0.02	0.12	0.86

In March the company was in the normal state.

Variant 8

Table 44

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.31	0.22	0.47
Pre-crisis	0.53	0.05	0.42
Crisis	0.06	0.12	0.82

In March the company was in the pre-crisis state.

Variant 9

Table 45

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.69	0.3	0.01
Pre-crisis	0.53	0.05	0.42
Crisis	0.1	0.12	0.78

In March the company was in the crisis state.

Variant 10

Table 46

The transition matrix

	Normal	Pre-crisis	Crisis
Normal	0.54	0.29	0.17
Pre-crisis	0.5	0.05	0.45
Crisis	0.18	0.24	0.58

In March the company was in the crisis state.

Absorbing Chains

A state of a Markov chain is called absorbing if it is impossible to leave it.

A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).

In an absorbing Markov chain, a state which is not absorbing is called transient.

The transition matrix will have the following canonical form:

$$\begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{c} \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \left(\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right) \end{array}$$

Here I is an identity matrix, 0 is a zero matrix, R is a matrix that describes relationships between transient and absorbing states, and Q is a matrix that describes relationships between transient states.

The theorem. In an absorbing Markov chain, the probability that the process will be absorbed is 1.

For an absorbing Markov chain the matrix $N = (E - Q)^{-1}$ is called the fundamental matrix for P. The entry N_{ij} of N gives the expected number of times that the process is in the transient state j if it is started in the transient state i.

We now consider the question: given that the chain starts in state i, what is the expected number of steps before the chain is absorbed? The answer is given in the next theorem.

The theorem. Let t_i be the expected number of steps before the chain is absorbed, given that the chain starts in state *i*, and let *t* be the column vector whose *i*-th entry is t_i .

Then $t = Nc$;

c is a column vector all of whose entries are 1.

The theorem. Let b_{ij} be the probability that an absorbing chain will be absorbed in the absorbing state *j* if it starts in the transient state *i*. Let *B* be the matrix with entries b_{ij}. Then *B* is an *t*-by-*r* matrix, and $B = NR$ *;* where *N* is the fundamental matrix and *R* is as in the canonical form.

Example 9. Some company may go bankrupt according to the two scenarios: Scenario 1 and Scenario 2. Before it goes bankrupt, the company may stay in one of possible transient states: Pre-bankruptcy 1 (PB1) or Pre-bankruptcy 2 (PB2). The state graph with appropriate one-step transition probabilities is shown in Fig. 50. One step equals one month.

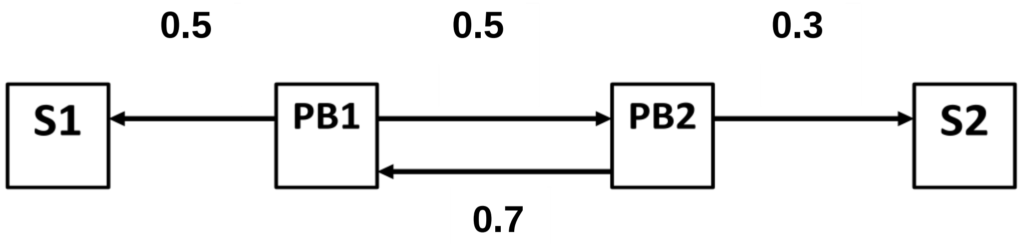


Fig. 50. **The graph of states**

Let state PB1 be the initial state of the company.

Determine the following:

1) the expected number of months in states PB1 and PB2 before the absorption;
2) for what initial conditions the company will be more often in state PB1;
3) the expected number of months before the company goes bankrupt;
4) under what initial conditions the company will go bankrupt faster;
5) the probabilities of both bankruptcy scenarios;
6) under what initial conditions the absorption probability of Scenario 2 will be the highest.

We form a Markov chain with states PB1, PB2, S1 and S2. States S1 and S2 are absorbing states. The transition matrix is shown in Fig. 51:

O17

	A	B	C	D	E	F
1		PB1	PB2	S1	S2	
2	PB1	0	0,5	0,5	0	
3	PB2	0,7	0	0	0,3	
4	S1	0	0	1	0	
5	S2	0	0	0	1	
6						

Fig. 51. **The transition matrix**

To select matrixes *Q* and *R*, insert the following formulas (Fig. 52).

	A	B	C	D	E	F	G	H
6								
7		PB1	PB2			S1	S2	
8		=B2	=C2	PB1		=D2	=E2	PB1
9	Q=	=B3	=C3	PB2	R=	=D3	=E3	PB2
10								

Fig. 52. **Matrixes *Q* and *R***

To calculate matrixes *N* and *B*, do the following.

For the **range** of cells B18:C19 choose MINVERSE, for the **range** of cells B22:C23 choose MMULT (Fig. 53).

	A	B	C
10			
11		1	0
12	E=	0	1
13			
14	E-Q=	=B11-B8	=C11-C8
15		=B12-B9	=C12-C9
16			
17			
18	N=	=MINVERSE(B14:C15)	
19			
20			
21			
22	B=	=MMULT(B18:C19;D2:E3)	
23			
24			

Fig. 53. **Calculating matrixes *N* and *B***

The results are shown in Fig. 54.

	A	B	C	D
16				
17		**PB1**	**PB2**	
18	N=	1,54	0,77	**PB1**
19		1,08	1,54	**PB2**
20				
21		**S1**	**S2**	
22	B=	0,77	0,23	**PB1**
23		0,54	0,46	**PB2**
24				

Fig. 54. **Final matrixes *N* and *B***

So, the answers to the questions are as follows:

1) if the initial state is PB1, then the expected number of times in transient states before the absorption are 1.54 for state PB1 and 0.77 for state PB2;

2) the company will be in state PB1 more often, if it starts in state PB1;

3) the expected number of months before the company goes bankrupt is 2.31 (if it starts in state PB1);

4) the company will go bankrupt faster if it starts in state PB1;

5) the probabilities of bankruptcy are 0.77 and 0.23 for Scenario 1 and Scenario 2, respectively;

6) the probability of Scenario 2 will be the highest for initial state PB2.

Tasks for self-study work

Some company may occur in four states: A, B, C and D. The state graph with appropriate one-step transition probabilities is shown below. One step equals one month.

Determine the following:

1) the expected number of months in transient states before the absorption;

2) under what initial conditions the company will be more often in each transient state?

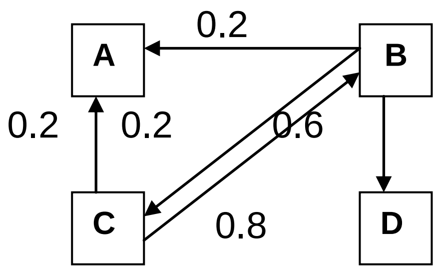
3) the expected number of months before the company is absorbed;

4) under what initial conditions the company will be absorbed faster;

5) probabilities of absorbing states;

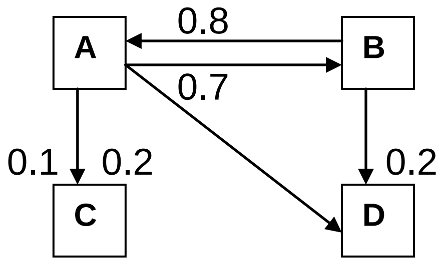
6) under what initial conditions the absorption probability of each absorbing state will be the lowest.

Variant 1



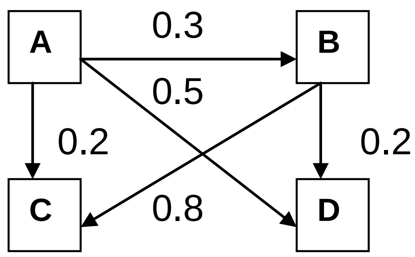
P(0) = (0, 1, 0, 0)

Variant 2



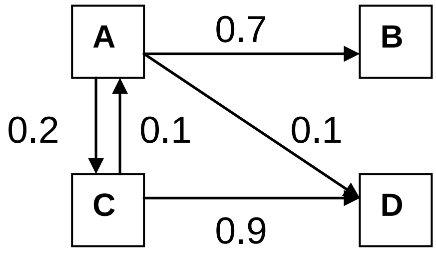
P(0) = (1, 0, 0, 0)

Variant 3



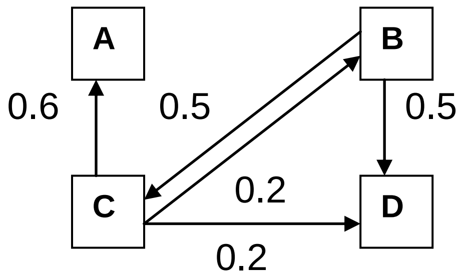
P(0) = (0,1, 0, 0)

Variant 4



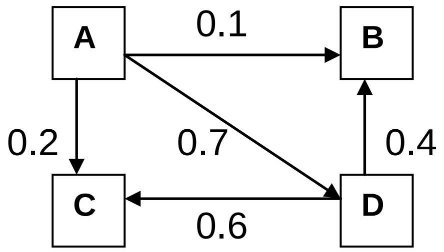
P(0) = (1, 0, 0, 0)

Variant 5



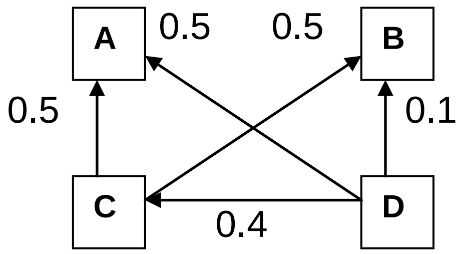
P(0) = (0, 0, 1, 0)

Variant 6



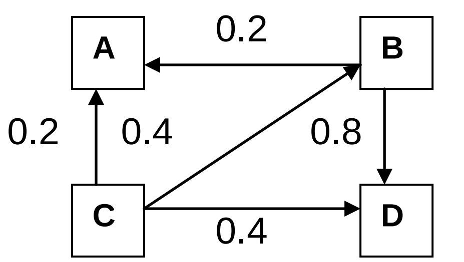
P(0) = (0, 0, 0, 1)

Variant 7



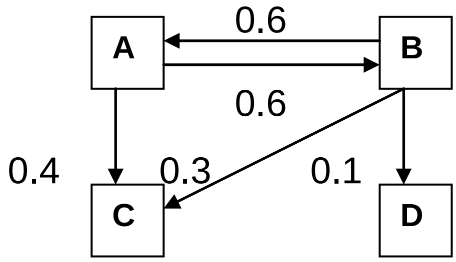
P(0) = (0, 0, 0, 1)

Variant 8



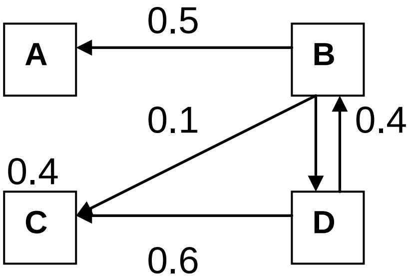
P(0) = (0, 0, 1, 0)

Variant 9



P(0) = (0, 0, 1, 0)

Variant 10



P(0) = (0, 0, 0, 1)

Theme 6. The game theory

The game theory describes the situations involving conflict in which the payoff is affected by the actions and counter-actions of intelligent opponents. Two-person zero-sum games play a central role in the development of the theory of games.

Example 10. Player 1 has two choices from which to select, and Player 2 has three alternatives for each choice of Player 1. The payoff matrix is given in Table 47.

Table 47

Payoff matrix T

	Player 2			
Player 1		j = 1	j = 2	j = 3
	i = 1	4	1	3
	i = 2	2	3	4

Determine the optimal strategies for both players.

A pure strategy pair (*ij*) is in equilibrium if and only if the corresponding element t_{ij} is both the largest in its column and the smallest in its row. Such an element is also called a saddle point, which represents a decision by two players upon which neither can improve by unilaterally departing from it.

Let's determine whether matrix *T* has a saddle point. Insert the following formulas as shown in Fig. 55.

	A	B	C	D	E	F	G	H
1		Player 2				min		
2	Player 1		j=1	j=2	j=3			maximin
3		i=1	4	1	3	=MIN(C3:E3)		=MAX(F3:F4)
4		i=2	2	3	4	=MIN(C4:E4)		
5	max		=MAX(C3:C4)	=MAX(D3:D4)	=MAX(E3:E4)			
6								
7	minmax		=MIN(C5:E5)					

Fig. 55. **Determining a saddle point**

The results are shown in Fig. 56.

	A	B	C	D	E	F	G	H
1		Player 2				min		
2	Play		j=1	j=2	j=3			maximin
3	er 1	i=	4	1	3	1		2
4		i=	2	3	4	2		
5	max		4	3	4			
6								
7	minmax		3					

Fig. 56. **The results of the calculations**

According to Fig. 56, Player 1 can assure the payoff to be $\max_i \min_j t_{ij} = 2$, while Player 2 plays in such a manner that Player 1 receives no more than $\min \max t_{ij} = 3$. The problem of how the difference $[\min_j \max_i t_{ij}] - [\max \min t_{ij}]$ should be subdivided between the players thus remains open. In such cases, the players naturally seek additional strategic opportunities to assure themselves of the largest possible share of this difference. To achieve this objective, they must select their strategies randomly to confuse each other.

Let's check the matrix for dominant strategies.

For Player 2 the second strategy ***dominates*** the third one because all the payoffs in the 2d column are less than the corresponding payoffs of the 3d column. Thus the third strategy must be excluded from the matrix (Table 48).

Table 48

Payoff matrix T

		Player 2	
		j = 1	j = 2
Player 1	i = 1	4	1
	i = 2	2	3

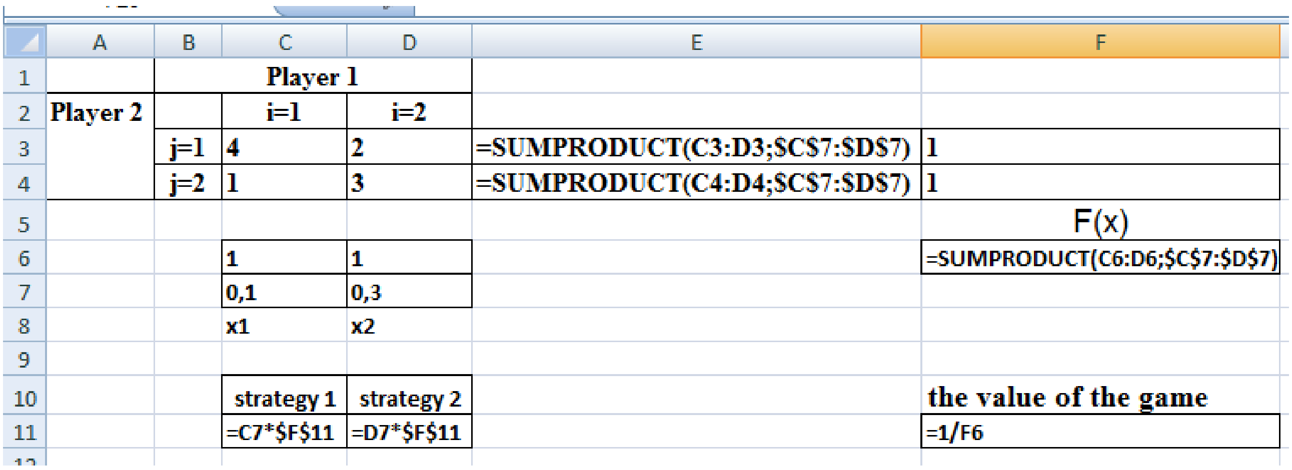
We will use a general method based on a linear programming formulation. The LP formulation for Player 1 is the following:

$4x_1 + 2x_2 \geq 1;$

$x_1 + 3x_2 \geq 1;$

$F(x) = x_1 + x_2 \rightarrow \min.$

The model we are going to solve looks as follows (Fig. 57).



	A	B	C	D	E	F
1			Player 1			
2	Player 2		i=1	i=2		
3		j=1	4	2	=SUMPRODUCT(C3:D3;$C$7:$D$7)	1
4		j=2	1	3	=SUMPRODUCT(C4:D4;$C$7:$D$7)	1
5						F(x)
6			1	1		=SUMPRODUCT(C6:D6;$C$7:$D$7)
7			0,1	0,3		
8			x1	x2		
9						
10			strategy 1	strategy 2		the value of the game
11			=C7*$F$11	=D7*$F$11		=1/F6

Fig. 57. **Player 1 – determining the mixed strategies**

Fill the solver parameters according to Fig. 58.

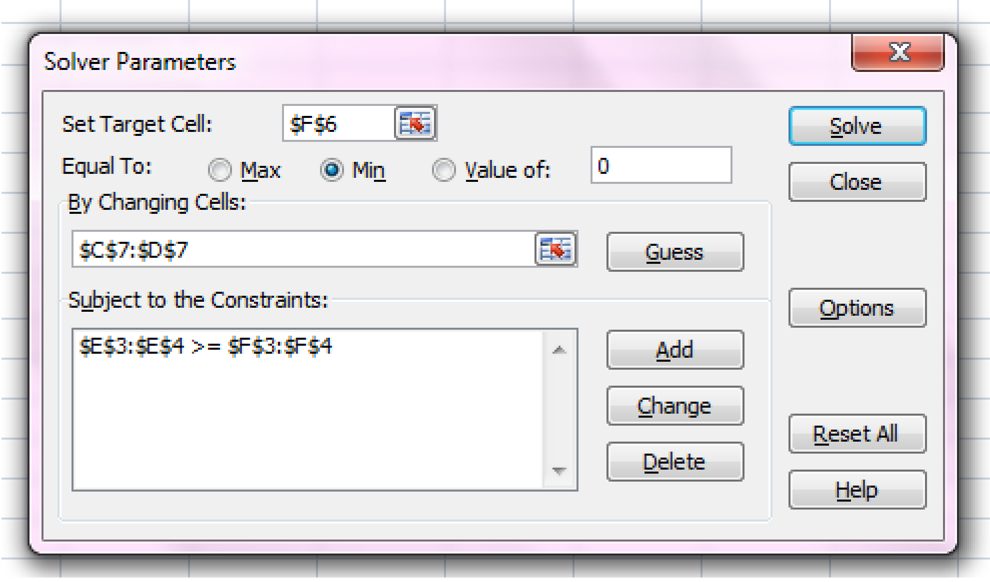
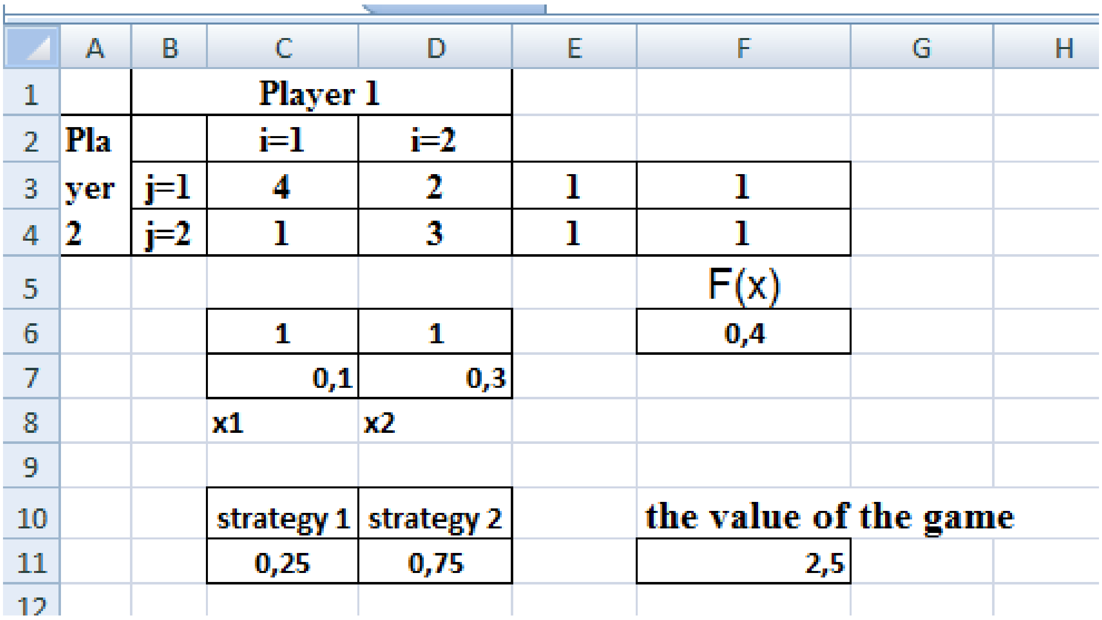


Fig. 58. **The Solver parameters**

The obtained results are shown in Fig. 59.



	A	B	C	D	E	F	G	H
1		Player 1						
2	Pla		i=1	i=2				
3	yer	j=1	4	2	1	1		
4	2	j=2	1	3	1	1		
5						F(x)		
6			1	1		0,4		
7			0,1	0,3				
8			x1	x2				
9								
10			strategy 1	strategy 2		the value of the game		
11			0,25	0,75		2,5		
12								

Fig. 59. **The mixed strategies for Player 1**

The LP formulation for Player 2:

$4y_1 + y_2 \le 1;$
$2y_1 + 3y_2 \le 1;$
$\Phi(y) = y_1 + y_2 \rightarrow \max.$

The model we are going to solve looks as follows (Fig. 60).

	A	B	C	D	E	F
1		Player 2				
2	Player 1		j=1	j=2		
3		i=1	4	1	=SUMPRODUCT(C3:D3;$C$7:$D$7)	1
4		i=2	2	3	=SUMPRODUCT(C4:D4;$C$7:$D$7)	1
5						
6			1	1		=SUMPRODUCT(C6:D6;$C$7:$D$7)
7			0,2	0,2		
8			y1	y2		
9						
10			strategy 1	strategy 2		the value of the game
11			=C7*$F$11	=D7*$F$11		=1/F6

Fig. 60. **Player 2 – Determining the mixed strategies**

Fill the solver parameters according to Fig. 61.

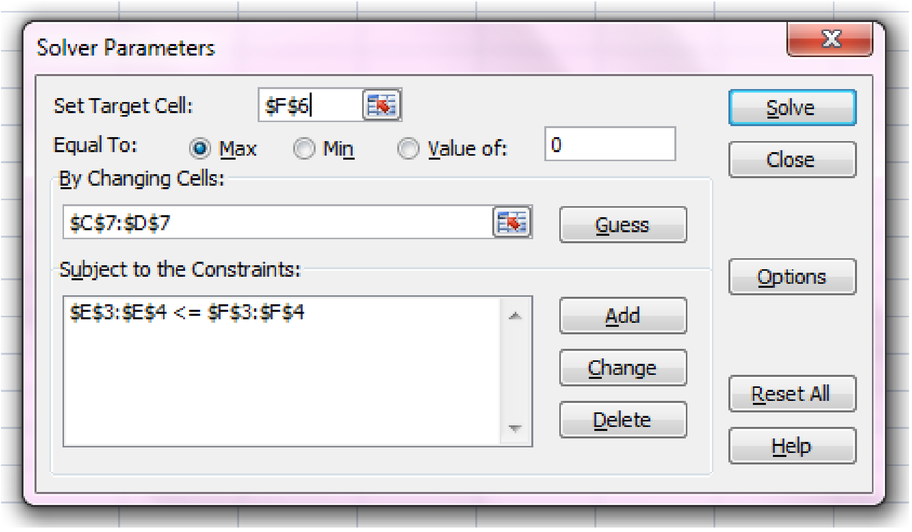


Fig. 61. **The Solver parameters**

The obtained results are shown in Fig. 62.

	A	B	C	D	E	F	G	H
1		Player 2						
2	Player 1		j=1	j=2				
3		i=1	4	1	1	1		
4		i=2	2	3	1	1		
5								
6			1	1		0,4		
7			0,2	0,2				
8			y1	y2				
9								
10			strategy 1	strategy 2		the value of the game		
11			0,5	0,5		2,5		
12								

Fig. 62. **The mixed strategies for Player 2**

The mixed saddle point is:

$q1 = 1/4$, $q2 = 3/4$; $p1 = 1/2$, $p2 = 1/2$, $p3 = 0$, and the value of the game equals 5/2.

Note that the essential strategies for Player 1 are $i = 1$, $i = 2$; for Player 2 they are $j = 1$, $j = 2$ while $j = 3$ is non-essential.

Tasks for self-study work

For the following payoff matrixes find a saddle point.

Variant 1

$$A = \begin{pmatrix} 4 & 5 & 6 & 7 & 9 \\ 3 & 4 & 6 & 5 & 6 \\ 7 & 6 & 10 & 8 & 11 \\ 8 & 5 & 4 & 7 & 3 \end{pmatrix}$$

Variant 2

$$A = \begin{pmatrix} 4 & 9 & 5 & 3 \\ 7 & 8 & 6 & 9 \\ 7 & 4 & 2 & 6 \\ 8 & 3 & 4 & 7 \end{pmatrix}$$

Variant 3

$$A = \begin{pmatrix} 9 & 8 \\ 6 & 9 \end{pmatrix}$$

Variant 4

$$A = \begin{pmatrix} 0.6 & 1 \\ 1 & 0.4 \end{pmatrix}$$

Variant 5

$$A = \begin{pmatrix} 3 & 5 & 1 & 2 \\ 4 & 2 & 4 & 3 \end{pmatrix}$$

Variant 6

$$A = \begin{pmatrix} 2 & 5 & 8 \\ 7 & 6 & 10 \\ 12 & 10 & 8 \end{pmatrix}$$

Variant 7

$$A = \begin{pmatrix} 3 & 1 & 3 & 7 \\ 1 & 9 & 3 & 0 \end{pmatrix}$$

Variant 8

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 1 & 0 \\ 0 & 2 & 5 \end{pmatrix}$$

Variant 9

$$A = \begin{pmatrix} 4 & 5 & 3 \\ 6 & 7 & 4 \\ 5 & 2 & 3 \end{pmatrix}$$

Variant 10

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 \\ 7 & 2 & 8 & 1 \end{pmatrix}$$

НАВЧАЛЬНЕ ВИДАННЯ

Методичні рекомендації до виконання лабораторних робіт з навчальної дисципліни "МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ В ЕКОНОМІЦІ ТА МЕНЕДЖМЕНТІ: ДОСЛІДЖЕННЯ ОПЕРАЦІЙ"

для студентів напряму підготовки 6.030601 "Менеджмент" денної форми навчання

Самостійне електронне текстове мережеве видання

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Редактор *З. В. Зобова*

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