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ABOUT SOLUTIONS' BEHAVIOR FOR SOME PDEs

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Abstract — **The aim of this brief communication is to is to study of the solutions' behavior for a wide class of nonlinear partial differential equations through the use of a new approach that has been proposed in [1].**

Key Terms **— partial differential equations, solutions' behavior.**

The mathematical formulation of the problem of given paper: to prove that the Cauchy problem for parabolic equation has shrinking property of support of the solutions. This is an important problem in terms of applied mathematics and mathematical physics. On order to achieve this goal the following tasks were solved:

to get integral estimates linking different norms of solution;

to reduce integral relationship to non-differential inequality and to analyze of this inequality;

to establish the property of shrinking of the support.

Let consider the problem

$$u_t-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-1}\frac{\partial u}{\partial x_i}\right)+|u|^{\lambda-1}u=0,\ t>0, \quad (1)$$

$$u(x,0)=u_0(x), \quad x\in\mathbb{R}^n, \quad (2)$$

We know that a problem has the instantaneous compactification property, if for any $t>0$ the support of the solution $u(x,t)$ is bounded even if it is unbounded for $t=0$. Main result of this brief communication is is the following theorem.

Theorem. In both of the cases

- $0<\lambda<1, \quad p\geq 1;$
- $0<\lambda<p,$

if $\dfrac{n-2}{n+2}<p<1$, when $n>2$, and in case $0<p<1$, when $n\leq 2$ the problem (1), (2) has the instantaneous compactification property.

Proof of the Theorem. For any numbers $0\leq\tau_1<\tau_2\leq T, \quad 0<s_1<s_2<\infty$, define by

$\Omega(s_1)=\{x\in R^n: |x|>s_1\}$;

$G_{\tau_1}^{\tau_2}(s_1)=\Omega(s_1)\times(\tau_1,\tau_2)$;

$K_{\tau_1}^{\tau_2}(s_1,\ s_2-s_1)=G_{\tau_1}^{\tau_2}(s_1)\setminus G_{\tau_1}^{\tau_2}(s_2)$.

Let us fix $\tau>0, s>0, \Delta\tau>0, \Delta s>0$ and $\eta(x,t)$ i $\eta_1(x)$: $\eta=1$ in $G_{\tau+\Delta\tau}^{T}(s+\Delta s)$; $\eta_1=1$ in $\Omega(s+\Delta s)$, $\eta=0$ in $R^n\times(0,T)\setminus G_{\tau}^{T}(s)$, $\eta_1=0$ in $\mathbb{R}^n\setminus\Omega(s)$. Suppose that $0\leq\eta_k\leq\dfrac{c}{\Delta s}$, $\left|\eta_{x_i}\right|\leq\dfrac{c}{\Delta s}$, $\left|\eta_{1x_i}\right|\leq\dfrac{c}{\Delta s}$; $\eta_k=0$ if $\tau+\Delta\tau<t<T$ and $\nabla\eta=0$ if $|x|>s+\Delta s$. As well-known an energy solution of (1), (2) is called the function such that $u(x,t)\in C\big((0,T);L_2(R^n)\big)\cap L_{1+p}\big((0,T);W_{p+1}^1(R^n)\big)\cap$ $\cap L_{\lambda+1}\big(R^n\times(0,T)\big)$ and satisfies identity (1) with $v\in L_{\lambda+1}\big(\mathbb{R}^n\times(0,T)\big)\cap W_{p+2,2}^{1,1}\big(\mathbb{R}^n\times(0,T)\big)$; (2)

$$\int_{\mathbb{R}^n}u(x,T_0)v(x,T_0)\,dx-\int_0^{T_0}\int_{\mathbb{R}^n}u(x,t)v_t(x,t)\,dxdt+$$
$$\int_0^{T_0}\int_{\mathbb{R}^n}\Big[|\nabla u|^{p-1}u_{x_i}v_{x_i}+|u|^{\lambda-1}uv\Big]dxdt=\int_{\mathbb{R}^n}u_0v(x,0)\,dx.$$

Note here, that the existence of solutions in the above sense is well known if $1\leq p$ and $0<\lambda\leq p$ see [2 4].

In order to proof the Theorem about compactification of solutions' support of the problem (1), (2) we need well-known Gagliardo-Nirenberg interpolation inequality, which will be given below, besides statement:

Lemma. If $f(\tau,s)$ – is positive, increasing function, which satisfies the inequality $f\left(\tau+f^{\alpha}\left(\tau,s\right),s+f^{\beta}\left(\tau,s\right)\right)\le\delta f\left(\tau,s\right)$ for each $\tau>\tau_0, s>s_0, \delta>1, \alpha>0, \beta>0$, then: $f\left(\tau,s\right)\equiv 0$ for all $\left(\tau,s\right)$ such that:
$\tau>\tau_0+\frac{1}{1-\delta^{\alpha}}f^{\alpha}\left(\tau_0,s_0\right)$,
$s>s_0+\frac{1}{1-\delta^{\beta}}f^{\beta}\left(\tau_0,s_0\right)$.

This Lemma that is not a trivial fact and therefore requires a strict mathematical proof which you can find in for example in [1]. So, let

$$E_T\left(\tau,s\right)=\int\limits_{G_\tau^T(s)}u^2dxdt,\ I_T\left(\tau,s\right)=\int\limits_{G_\tau^T(s)}|u|^{p+1}dxdt.$$

If we show that for $\forall\tau>0\ \exists s\left(\tau\right)<\infty$:

$H=H_T\left(\tau,s\right):=E_T\left(\tau,s\right)+I_T\left(\tau,s\right)=0$, then (thanks to the Lemma) we will obtain the Theorem. Thus, it is enough to show: $H_T\left(0,s\right)\to 0,\quad s\to\infty$,
$H\left(\tau+H^{\alpha},s+H^{\beta}\right)\le\mu H, \alpha>0, \beta>0, 0<\mu<1.$

Let substitute $v=u\eta^{p+1}$ into integral identity and integrating by parts

$$\frac{1}{2}\int\limits_{R^n}u^2\left(x,T\right)\eta^{p+1}\left(x,T\right)dx+\int\limits_0^T\int\limits_{R^n}|\nabla u|^{p+1}\eta^{p+1}dxdt$$
$$+\int\limits_0^T\int\limits_{R^n}|u|^{\lambda+1}\eta^{p+1}dxdt=(p+1)\int\limits_0^T\int\limits_{R^n}\frac{1}{2}u^2\eta_t\eta^p dxdt+$$
$$\int\limits_0^T\int\limits_{R^n}|\nabla u|^{p-1}u_{x_i}u\eta_{x_i}\eta^p dxdt \qquad (4)$$

For the right-hand side of (4) we apply Young's inequality with ε:

$$\int\limits_{\Omega(s)}u^2\eta^{p+1}dx+\int\limits_{G_\tau^T(s)}\left(|\nabla u|^{p+1}+|u|^{\lambda+1}\right)\eta^{p+1}dxdt\le$$
$$\le c\left[I_T+E_T\right]:=cR_1. \qquad (5)$$

Let us Gagliardo-Nirenberg inequality use under $\alpha=2,\ \beta=p+1,\ \gamma=\lambda+1$:

$$\|v\|_{\alpha,\Omega(s)}\le d_1\|\nabla v\|_{\beta,\Omega(s)}^{\Theta}\|v\|_{\gamma}^{1-\Theta},$$

where

$\frac{1}{\alpha}=\Theta\left(\frac{1}{\beta}-\frac{1}{n}\right)+\left(1-\Theta\right)\frac{1}{\gamma},\ \gamma>1, \beta>1$ and involve Young's inequality:

$$\left(\int\limits_{\Omega(\bar{s})}u^2dx\right)^{1-\nu}\le c\int\limits_{\Omega(\bar{s})}\left(|\nabla u|^{p+1}+|u|\lambda+1\right)dx,$$

with $\bar{s}>s_0>0, \nu=\frac{\left(p+1\right)\left(1-\lambda\right)}{2\left(p+1\right)+n\left(p-\lambda\right)}<1.$

Integration leads to the inequality:

$$\Psi_{\bar{\tau},\bar{s}}^T\left(1-\nu\right):=\int_{\bar{\tau}}^T\left(\int\limits_{\Omega(\bar{s})}u^2dx\right)^{1-\nu}dt\le$$
$$\le c\int\limits_{G_\tau^T(\bar{s})}\left(|\nabla u|^{p+1}+|u|^{\lambda+1}\right)dxdt.$$

We return back to the integral identity with test function $v=u\eta^{p+1}\chi_l\left(t\right),\ l>0,$

$\chi_l\left(t\right)=\int\limits_0^t\left(\int\limits_{\Omega(s)}u^2\eta^{p+1}dx\right)^l dt$ and obtain:

$$\chi_{l+1}\left(T\right)=\chi_l\left(T\right)\int\limits_{\Omega(s)}u^2\eta^{p+1}dx+$$
$$+\int\limits_{G_\tau^T(s)}\left[2|u|^{\lambda+1}\eta^{p+1}-u^2\left(\eta^{p+1}\right)_t\right]\chi_l\left(t\right)dxdt+$$
$$+\int\limits_{G_\tau^T(s)}\left[2|\nabla u|^{p-1}u_{x_i}\left(u\eta^{p+1}\right)_{x_i}\right]\chi_l\left(t\right)dxdt,$$

from which and (5) we have: $\chi_l\left(T\right)\le c\chi_\delta\left(T\right)R_1^{l-\delta}$ for some $l>\delta>0$. According to the definition of $\eta\left(x,t\right)$ and previous compute, we obtain several inequalities, which are crucial:

$$\Psi_{\tau+\Delta\tau,s+\Delta s}^T\left(l\right)\le\chi_l\left(T\right)\le\Psi_{\tau,s}^T\left(l\right),$$
$$\Psi_{\tau,s}^T\left(1-\nu\right)\le cR_1\left(s,\Delta s,\tau,\Delta\tau\right), \qquad (6)$$
$$\chi_1\left(T\right)\le c\chi_{1-\nu}\left(T\right)R_1^{\nu}\left(s,\Delta s,\tau,\Delta\tau\right), \qquad (7)$$
$$\chi_{1-\nu}\left(T\right)\le\Psi_{\tau,s}^T\left(1-\nu\right). \qquad (8)$$

By the definition of energy function E_T :(5)

$$\Psi_{\tau+\Delta\tau,s+\Delta s}^T\left(1\right):=E_T\left(\tau+\Delta\tau,s+\Delta s\right)\le\chi_1\left(T\right) \quad (9)$$

Substitute (7) into (9) and using (8), (6) we obtain that

$$E_T(\tau+\Delta\tau, s+\Delta s) \le cR_1^{1+\nu}(s,\Delta s,\tau,\Delta\tau). \quad (10)$$

Starting from this place we should distinguish three possible cases p takes.

In the ***case*** $p=1$ we have identity $I_T(\tau,s)=E_T(\tau,s)$ and proof is trivial, because it is immediately follows $\forall\tau>0\,\exists\, s(\tau)<\infty : H=H_T(\tau,s):=E_T(\tau,s)+$ $+I_T(\tau,s)=2\cdot E_T(\tau,s)$, thus, by (10) and thanks to Lemma we have result of Theorem.

Case $p>1$. Put into integral identity $\alpha=p+1, \beta=p+1, \gamma=2$. After integrating in t, using the Holder inequality

$$I_T(\tau+\Delta\tau,s+\Delta s)\le c\left(\int\limits_{G^T_{\tau+\Delta\tau}(s+\Delta s)}|\nabla u|^{p+1}\,dxdt\right)^{\theta_1}\cdot$$

$$\left(\Psi^T_{\tau+\Delta\tau,s+\Delta s}\left(\frac{p+1}{2}\right)\right)^{1-\theta_1}, \quad (11)$$

here $\theta_1=\dfrac{n(p-1)}{2(p+1)+n(p-1)}<1$. Inequalities (6) (8) under $l=\dfrac{1+p}{2}$ and $\delta=1-\nu$ lead to the following correlation

$$\Psi^T_{\tau+\Delta\tau,s+\Delta s}\left(\frac{p+1}{2}\right)\le c\Psi^T_{\tau,s}(1-\nu)\,R_1^{\frac{1+p}{2-1+\nu}}.$$

Using result of (10) to the last estimate we obtain: $\Psi^T_{\tau+\Delta\tau,s+\Delta s}\left(\dfrac{p+1}{2}\right)\le cR_1^{\frac{1+p}{2+\nu}}$.

We apply the last inequality to ratio (11), so $I_T(\tau+\Delta\tau,s+\Delta s)\le cR_1^{1+\nu_1}$,

$$\nu_1=(1-\theta_1)\left(\frac{p-1}{2}+\nu\right)=\frac{\nu(p-\lambda)}{1-\lambda}>\nu. \quad (12)$$

Now add (10) and (12), use definition of the function R_1,

$$H_T(\tau+\Delta\tau,s+\Delta s)\le$$

$$\le c_0\Delta_\tau E_T(\tau,s)\left\{\frac{(\Delta_\tau E_T(\tau,s))^{\nu}}{(\Delta\tau)^{1+\nu}}+\frac{(\Delta_\tau E_T(\tau,s))^{\nu_1}}{(\Delta\tau)^{1+\nu_1}}\right\}$$

$$+c_0\Delta_s I_T(\tau,s)\left\{\frac{(\Delta_s I_T(\tau,s))^{\nu_1}}{(\Delta s)^{(1+p)(1+\nu_1)}}+\frac{(\Delta_s I_T(\tau,s))^{\nu}}{(\Delta s)^{(1+p)(1+\nu)}}\right\}$$

where

$$\Delta_\tau f(\tau,s):=f(\tau,s)-f(\tau+\Delta\tau,s),$$
$$\Delta_s f(t,s):=f(\tau,s)-f(\tau,s+\Delta s).$$

Now let us fix $\Delta s=(I_T(\tau,s))^{\frac{\nu}{(p+1)(\nu+1)}}$, $\Delta\tau=(E_T(\tau,s))^{\frac{\nu}{1+\nu}}$. As E and I are monotone, we come to inequality

$$H_T\left(\tau+H_T^{\frac{\nu}{1+\nu}}(\tau,s,),s+H_T^{\frac{\nu}{(1+p)(1+\nu)}}(\tau,s)\right)\le$$

$$\le\mu_1 H_T(\tau,s). \quad (13)$$

In ***case*** $0<p<1$ it is easy (using the same approach) to obtain an inequality analogous to (13), which is to complete series of compute of our proof, but, of course, with other index, namely,

$$\nu_1=\frac{\nu(p-\lambda)}{1-\lambda}<\nu.$$

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