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SIMON KUZNETS KHARKIV NATIONAL UNIVERSITY OF ECONOMICS

HIGHER AND APPLIED MATHEMATICS

Guidelines to practical tasks
to the part "Applied Mathematics"
for Bachelor's (first) degree students
of speciality 242 "Tourism"

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The sufficient theoretical material on the academic discipline and typical examples are presented to help students master the material of the part "Applied Mathematics" and apply the obtained knowledge to practice. Tasks for independent work and individual tasks, a list of theoretical questions are given to promote the improvement and extension of students' knowledge of this part.

For Bachelor's (first) degree students of speciality 242 "Tourism".

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Introduction

Probability theory is the part of mathematics that aims to provide insight into phenomena that depend on chance or on uncertainty. The most prevalent use of the theory comes through the frequentists' interpretation of probability in terms of the outcomes of repeated experiments, but probability is also used to provide a measure of subjective beliefs, especially as judged by one's willingness to place bets. If we want to predict the chance of something happening in the future, we use probability.

Probability theory plays an important role in everyday life in economics, in business, in trade on financial markets, in risk assessment and many other areas where statistics is applied to the real world.

Owing to the study of probability theory a student is obliged to receive the basic knowledge of this part and use skills of applying the elements of probability theory in investigations where probability theory is applied as an instrument of investigation for forming economic mathematical models of economic processes and developments. This makes it possible for him to apply the acquired knowledge and skills for solving many practical problems of economics and business.

Guidelines to applied mathematics

Theme 11. Empirical and logical bases of probability theory

Bases of probability theory

Let's consider the *fundamental concepts* of probability theory.

An ***experiment*** is a repeatable process that gives rise to a number of outcomes.

An ***outcome*** is something that follows as a result or consequence.

An ***event*** is a collection (or set) of *one or more* outcomes.

Events are sets and set notation is used to describe them. We use upper case letters to denote events. They are denoted as $A, B, C, \ldots, A_1, A_2, \ldots$

The simplest indivisible mutually exclusive outcomes of an experiment are called ***elementary events*** $\omega_1, \omega_2, \ldots$

A ***space of elementary events (sample space)*** is the set of all possible elementary outcomes of an experiment, which is denoted by Ω.

Any subset of Ω is called a ***random event*** A (or simply an ***event*** A).

Elementary events that belong to A are said to ***favor*** A.

An event is ***certain*** (or ***sure***) if it always happens.

An event is **impossible** if it never happens.

Equally likely events are such events that have the equal chance to happen in an experiment.

Example 11.1. The experiment (tossing a coin once) has 2 outcomes: head (the first outcome) and tail (the second outcome). The event A is getting "head". For this experiment the sample space is $\Omega = \{head, tail\}$.

The mathematics of probability is expressed most naturally in terms of sets, therefore, let's consider *basic operations with events*.

The ***intersection*** $C = A \cap B = A \cdot B$ of events A and B is the event where both A and B occur. The elementary outcomes of the intersection $A \cdot B$ are the elementary outcomes that simultaneously belong to A and B.

Example 11.2. If $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ are given, then $C = A \cap B = \{1, 3\}$.

When events A and B have no outcomes in common ($A \cap B = \varnothing$ (this symbol $\varnothing$ is called the empty set)), they are ***mutually exclusive*** (or ***incompatible***) ***events***.

Example 11.3. If events $A = \{1, 2\}$ and $B = \{3, 5\}$ are given, then $C = A \cap B = \varnothing$, because events A and B have no outcomes in common.

When events A and B have common outcomes ($A \cap B \neq \varnothing$), they are ***not mutually exclusive*** (or ***compatible***) ***events***.

Example 11.4. In the experiment of throwing a dice the event A of getting an odd number ($A = \{1, 3, 5\}$) and the event B of getting a number greater than 3 ($B = \{4, 5, 6\}$) are not mutually exclusive, i.e. they are compatible, because $A \cap B = \{5\} \neq \varnothing$.

The ***union*** $C = A \cup B = A + B$ of events A and B is the event that at least one of the events A or B occurs. The elementary outcomes of the union $A + B$ are the elementary outcomes that belong to at least one of the events A and B.

Example 11.5. If events $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6\}$ are given, then $C = A \cup B = \{1, 2, 3, 4, 5, 6\}$.

Two events A and $\overline{A}$ are said to be ***opposite*** (***complementary***) if they simultaneously satisfy the following conditions: $A \cup \overline{A} = \Omega$ and $A \cap \overline{A} = \varnothing$.

The ***difference*** $C = A \setminus B = A - B$ of events A and B is the event "A occurs and B does not occur". The elementary outcomes of the difference $A \setminus B$ are the elementary outcomes of A that do not belong to B.

Example 11.6. If events $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5\}$ are given, then $C = A \setminus B = \{2, 4\}$.

An event A ***implies*** an event B ($A \subset B$) if B occurs in each realization of an experiment for which A occurs.

Example 11.7. If events $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5\}$ are given, then the event A implies the event B or $A \subset B$.

Events A and B are said to be ***equivalent*** ($A = B$) if A implies B ($A \subset B$) and B implies A ($B \subset A$), i.e., if, for each realization of an experiment, both events A and B occur or do not occur simultaneously.

Example 11.8. If events $A = \{1, 2, 3\}$ and $B = \{3, 2, 1\}$ are given, then events A and B are equivalent or $A = B$.

Venn diagrams are useful for visualizing the relationships between sets or events (Fig. 11.1).

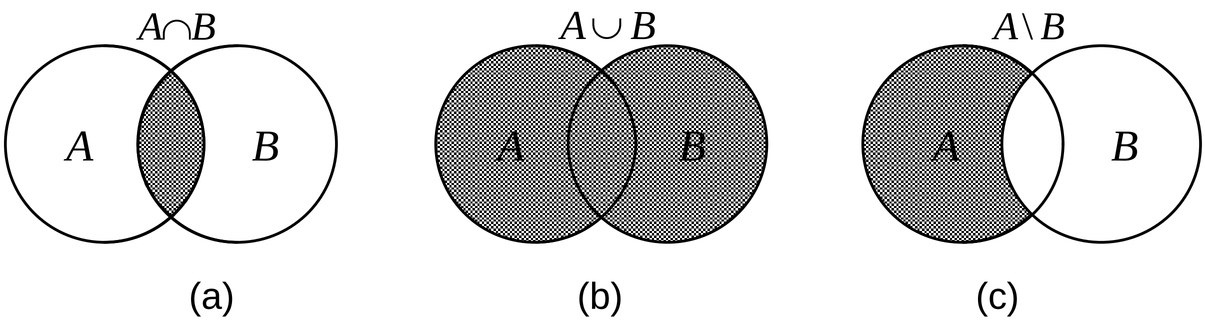

Fig. 11.1. **The intersection $A \cap B$ (a), the union $A \cup B$ (b) and the difference $A \setminus B$ (c) of events A and B**

When solving problems of probability theory one can use the *following rules*:

1. The ***rule of sum*** is an intuitive principle stating that if there are a possible outcomes for an event A (or ways to do something) and b possible outcomes for another event B (or ways to do another thing) and two events can't occur together (or the two things can't be done) (A and B are mutually exclusive or incompatible events) then there are $a + b$ total possible outcomes for the events A and B (or total ways to do one of the things); formally, the sum of sizes of two incompatible sets is equal to the size of their union, i.e. $|A| + |B| = |A \cup B|$.

Example 11.9. A woman has decided to shop at one store today, either in the north part of town or the south part of town. If she visits the north part of town, she will either shop at a mall, a furniture store, or a jewelry store (3 ways). If she visits the south part of town then she will either shop at a clothing store or a shoe store (2 ways).

Solution. Let A be the woman visiting the north part of town and B be the woman visiting the south part of town, i.e. $|A| = 3$ and $|B| = 2$. Thus there are $|A \cup B| = |A| + |B| = 3 + 2 = 5$ possible shops the woman could end up shopping at today.

2. The ***rule of product*** is another intuitive principle stating that if there are a possible outcomes for an event A (or ways of doing something) and b possible outcomes for another event B (or ways of doing another thing) and both events can occur at the same time (or the two things can be done) (A and B are not mutually exclusive or compatible events) then there are $a \cdot b$ total ways of performing both things, i.e. $|A \cap B| = |A| \cdot |B|$.

Example 11.10. When we decide to order pizza, we must first choose the type of crust: thin or deep dish (2 choices or $|A| = 2$). Next, we choose the topping: cheese, pepperoni, or sausage (3 choices or $|B| = 3$).

Solution. Using the rule of product, you know that there are $|A \cap B| = |A| \cdot |B| = 2 \cdot 3 = 6$ possible combinations of ordering a pizza.

3. ***The inclusion-exclusion principle*** or the *rule of inclusion and exclusion*: the inclusion-exclusion principle relates to the size of the union of multiple sets, the size of each set and the size of each possible intersection of the sets. The smallest example is when there are two sets: the number of elements in the union of the events A and B is equal to the sum of the elements in the events A and B minus the number of elements in their intersection, i.e. $|A \cup B| = |A| + |B| - |A \cap B|$.

Example 11.11. 35 voters were queried about their opinions regarding two referendums. 14 supported referendum 1 and 26 supported referendum 2. How many voters supported both, assuming that every voter supported either referendum 1 or referendum 2 or both?

Solution. Let A be voters who supported referendum 1 and B be voters who supported referendum 2. Then we have $|A \cup B| = 35$, $|A| = 14$ and $|B| = 26$. Using the inclusion-exclusion principle we obtain:

$$|A \cup B| = |A| + |B| - |A \cap B| \quad \text{or} \quad |A \cap B| = |A| + |B| - |A \cup B|$$
$$|A \cap B| = 14 + 26 - 35 \quad \text{or} \quad |A \cap B| = 5.$$

The classical definition and calculation of probability

Let a space of elementary events Ω be given and this space consist of n equally likely elementary outcomes (i.e. the total number of outcomes) of the experiment, among which there are m outcomes, favorable for an event A (i.e. a number of outcomes of the event A can happen), and $\Omega \subset A$. Then the number:

$$P(A) = \frac{m}{n} \tag{11.1}$$

is called the ***probability of the event*** A.

As all events have probabilities between impossible (0) and certain (1), then probabilities are usually written as a fraction, a decimal or sometimes as a percentage. In this lecture probabilities will be written as fractions or decimals.

The probability is the non-dimensional quantity. It can be measured in percent from 0 to 100. For example, $P(A) = \frac{4}{10} = 0.4 = 40\ \%$.

Example 11.12. Suppose the event A we are going to consider is *rolling a die* once and obtaining a 3.

Solution. The die could land in a total of six different ways. We say that the total number n of outcomes of rolling the die is six, which means there are six ways it could land. The number m of ways of obtaining the particular outcome of A is one.

We can apply formula (11.1) and find: $P(A) = \frac{1}{6}$.

When we roll a die, it has an equal chance of landing on any of the six numbers 1, 2, 3, 4, 5, or 6. These events are called equally likely events.

Basic formulas of combinatorics

We often compose new sets, systems or sequences from the elements of a given set in a certain way. Depending on the way we do it, we get the notion of permutation, combination and arrangement. The basic problem of combinatorics is to determine how many different choices or arrangements are possible with the given elements (for instance, letters of an alphabet, books of a library, cars on a parking, etc.).

Collection of formulas of combinatorics without repetitions

A ***permutation without repetitions*** is a number of different permutations of n different elements:

$$P_n = 1 \cdot 2 \cdot \ldots \cdot n = n!$$

Example 11.13. In a classroom, 16 students are seated in 16 places. There are $P_{16} = 16!$ different possible permutations.

An ***arrangement without repetitions*** is an ordering of k elements selected from n different ones, i.e. arrangements are combinations considering the order:

$$A_n^k = \frac{n!}{(n-k)!}.$$

Example 11.14. How many different ways are there to choose a chairman, his deputy, and a first and a second assistant for them out of 30 participants at an election meeting?

Solution. This answer is $A_{30}^4 = \frac{30!}{(30-4)!} = 657720$.

A ***combination without repetitions*** is called a choice of k elements from n different elements without considering the order of them:

$$C_n^k = \frac{n!}{k! \cdot (n-k)!}$$

Example 11.15. There are $C_{30}^4 = \frac{30!}{4! \cdot (30-4)!} = 27405$ possibilities to choose an electoral board of 4 persons out of 30 participants.

Example 11.16. 7 tickets were distributed among 17 students including 8 girls. What is the probability that there are 4 girls among ticket owners?

Solution. The number of possible ways of distributing 7 tickets among 17 students is equal to the number of combinations of 17 elements taken 7 at a time, i.e. C_{17}^7. The number of the selection of 4 girls from 8 is C_8^4. Each group of 4 can be connected with each group of 3 of 9 boys. The number of such groups of 3 is C_9^3. The number of the results of distributing 7 tickets including 4 tickets for girls and 3 for boys, is $C_8^4 \cdot C_9^3$. Then the probability is:

$$P(A) = \frac{C_8^4 \cdot C_9^3}{C_{17}^7}.$$

Collection of the formulas of combinatorics with repetitions

If for different elements k out of n elements with replacement, no subsequent ordering is performed (i.e., each of the n elements can occur 1, 2, ..., or k times in any combination), then one speaks of ***combinations with repetitions***. The number $\overline{C}_n^k$ of all distinct combinations with repetitions of n elements taken k at a time is given by the formula:

$$\overline{C}_n^k = C_{n+k-1}^k.$$

Example 11.17. Consider the set of elements 1, 2, 3 ($n = 3$). Take $k = 2$ elements, there are $\overline{C}_3^2 = C_{3+2-1}^2 = C_4^2 = \frac{4!}{2! \cdot (4-2)!} = 6$ combinations with repetitions $[(1, 2), (1, 3), (2, 3), (1, 1), (2, 2), (3, 3)]$.

If for different elements k out of n elements with replacement, the chosen elements are ordered in some way, then one speaks of ***arrangements with repetitions***. The number $\overline{A}_n^k$ of distinct arrangements with repetitions of n elements taken k at a time is given by the formula:

$$\overline{A}_n^k = n^k.$$

Example 11.18. Consider the set of elements 1, 2, 3 ($n = 3$). Take $k = 2$ elements, there are $\overline{A}_3^2 = 3^2 = 9$ arrangements with repetitions $[(1, 2),$ $(1, 3), (2, 3), (2, 1), (3, 1), (3, 2), (1, 1), (2, 2), (3, 3)]$.

Let's suppose that a set of n elements contains k distinct elements, of which the first occurs n_1 times, the second occurs n_2 times, ..., and the k-th occurs n_k times, $n_1 + n_2 + \ldots + n_k = n$. Permutations of n elements of this set are called ***permutations with repetitions on*** n ***elements***. The number $P_n(n_1, n_2, \ldots, n_k)$ of permutations with repetitions on n elements is given by the formula:

$$P_n(n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}.$$

Example 11.19. If there are two letters a and one letter b, the number of permutations with repetitions out of 3 elements and composition of letters 2, 1 equals $P_3(2, 1) = \frac{3!}{2! \cdot 1!} = 3$ $[(a, a, b), (a, b, a), (b, a, a)]$.

A statistical definition of a probability of an event A

Let A be an event belonging to the sample space Ω, i.e. $A \supset \Omega$, of an experiment. If the event A occurred n_A times while we repeated the experiment n times, then n_A is called the ***frequency***, and $\frac{n_A}{n}$ is called the relative frequency of the event A.

A geometric definition of a probability of an event A

Let Ω be a set of a positive finite measure $\mu(\Omega)$ and consist of all measurable (i.e. having a measure) subsets $A \supset \Omega$. The ***geometric probability*** of an event A is defined to be the ratio of the measure of A to that of Ω, i.e.

$$P(A) = \frac{\mu(A)}{\mu(\Omega)}.$$

The notion of geometric probability is not invariant under transformations of the sample space Ω and depends on how the measure $\mu(A)$ is introduced. Different geometric measures, for example, lengths, areas or volumes can be used as measures $\mu(A)$ and $\mu(\Omega)$.

Example 11.20. A point is randomly thrown into a disk of radius $R=1$. Find the probability of the event that the point lands in the disk of radius $r = \frac{1}{2}$ centered at the same point.

Solution. Let A be the event that the point lands in the smaller disk.

We find the probability $P(A)$ as the ratio of the area of the smaller disk to that of the larger disk:

$$P(A) = \frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2} = \frac{(1/2)^2}{1^2} = \frac{1}{4}.$$

Types of events and their properties

An event A is said to be ***impossible*** if it cannot occur for any realization of the experiment. Obviously, the impossible event does not contain any elementary outcome and hence should be denoted by the symbol Ø. Its probability is zero, i.e. $P(A) = 0$.

Example 11.21. Let's roll a die and obtain a score of 7 (the event A). It's an impossible event, then $P(A) = 0$.

Property 1. The probability of an impossible event is 0, i.e. $P(A) = 0$.

An event A is said to be ***sure*** if it is equivalent to the space of elementary events Ω, i.e. $A=\Omega$, or it happens with probability 1.

Example 11.22. Let's roll dice and obtain a score less than 13 (the event A). It's a sure event or a space of elementary events Ω, because it consists of all possible outcomes of Ω. Then $P(A)=1$.

Property 2. The probability of a sure (certain) event is 1, i.e. $P(A)=1$.

Property 3. The probability of a space of elementary events Ω is 1, i.e. $P(\Omega)=1$.

Property 4. All probabilities that lie between zero and one are inclusive, i.e. $0 \le P(A) \le 1$.

The event where A doesn't occur is called the ***complement*** of A, or the ***complementary event***, and is denoted by $\overline{A}$. The elementary outcomes of $\overline{A}$ are the elementary outcomes that don't belong to the event A.

Property 5. The probability of the event $\overline{A}$ opposite to the event A is equal to $P(\overline{A})=1-P(A)$.

From this property we can obtain that $P(A)+P(\overline{A})=1$ for complementary events $\overline{A}$ and A and explain them in the next example.

Example 11.23. Helen rolls a die once. What is the probability she rolls an even number or an odd number?

Solution. The event of rolling an even number (A) and the event of rolling an odd number (B) are mutually exclusive events, because they cannot happen at the same time, so we add the probabilities. In addition, these two events make up all the possible outcomes, so they are complementary events, i.e. B is $\overline{A}$. Let's write: $P(A)+P(B)=\frac{3}{6}+\frac{3}{6}=1$.

The events A and B are called ***equally likely events*** if $P(A)=P(B)$.

Property 6. Probabilities of equally likely events A and B are equal, i.e. $P(A)=P(B)$.

Example 11.24. When we roll a die it has an equal chance $\frac{1}{6}$ of landing on any of the six numbers 1, 2, 3, 4, 5, or 6. These events are called equally likely events.

Property 7. Nonnegativity: $P(A)\ge 0$ for any $A\subset\Omega$.

Property 8. For each $A \subset \Omega$ the inequality $P(A) \leq 1$ takes place.

Property 9. If an event A implies B, i.e. $A \subset B$, then $P(A) \leq P(B)$.

Theme 12. Basic theorems of probability theory, their economic meaning

Addition theorems of probabilities

The events are called ***compatible*** (***joint or mutually nonexclusive***) if they can occur together in the same experiment.

The events are called ***incompatible*** (***disjoint or mutually exclusive***) if they cannot occur together in the same experiment.

The probability addition theorem for incompatible events. The probability of realization of at least one of two events A and B is given by the formula:

$$P(A+B) = P(A\, or\ B) = P(A) + P(B), \tag{12.1}$$

where A and B are incompatible events.

The probability of such events is explained in the following example.

Example 12.1. Ann rolls a die once. a) What is the probability she rolls a 3 and a 6? b) What is the probability she rolls a 3 or a 6?

Solution. a) When one die is rolled, the event A of rolling a 3 and the event B of rolling a 6 are events that cannot both happen at the same time, and are called mutually exclusive events. So the probability of rolling a 3 and a 6 is impossible on one roll of a die, and equal to zero, i.e. $P(A\, and\, B) = 0$.

b) The probability of rolling a 3 (A) or a 6 (B) is also a mutually exclusive event and is calculated by formula (12.1):

$$P(A+B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

The probability addition theorem for compatible events. The probability of realization of at least one of two events A and B is given by the formula

$$P(A+B)=P(A\,or\,B)=P(A)+P(B)-P(A\cap B), \qquad (12.2)$$

where A and B are compatible events.

Example 12.2. Ann rolls a die once. What is the probability she rolls a prime number or an odd number?

Solution. When one die is rolled, the event A of rolling a prime or the event B of rolling an odd number are events that can both happen at the same time, and they are compatible events. Then the probability of A or B is calculated by formula (12.2).

We have $A=\{2,3,5\}$, $B=\{1,3,5\}$ and obtain $P(A)=\frac{3}{6}$, $P(B)=\frac{3}{6}$.

We find $A\cap B=\{3,5\}$, $P(A\cap B)=\frac{2}{6}$ and use formula (12.2):

$$P(A+B)=P(A)+P(B)-P(A\cap B)=\frac{3}{6}+\frac{3}{6}-\frac{2}{6}=\frac{4}{6}=\frac{2}{3}.$$

Multiplication theorems of probabilities

The events are called ***independent*** if the occurrence of one of them does not change the probability of the occurrence of the other one.

The events are called ***dependent*** if the probability of each of them is changed in connection with the occurrence or nonoccurrence of the other one.

The multiplication theorem for independent events. When the outcome of one event has no effect on the outcome of another event, we say that the two events are independent events. To obtain the probability of independent events we multiply the probabilities of the separate events, i.e.

$$P(A\cdot B)=P(A\,and\,B)=P(A)\cdot P(B), \qquad (12.3)$$

where A and B are independent events.

Example 12.3. A coin is tossed and a die is rolled. What is the probability of obtaining a head and a prime number?

Solution. The result of tossing a coin cannot possibly affect the outcome of rolling a die. In other words, if the coin landed as a head, it would not affect the way the die would land. Then the outcomes are independent events.

The probability of A (tossing a head) is $\frac{1}{2}$, i.e. $P(A)=\frac{1}{2}$, and the probability of B (rolling a prime number with a die) is $\frac{3}{6}$, i.e. $P(B)=\frac{3}{6}$, because there are three numbers 2, 3, and 5 that are prime. Let's use formula (12.3): $P(A\cdot B)=P(A)\cdot P(B)=\frac{1}{2}\cdot\frac{3}{6}=\frac{1}{4}$.

The multiplication theorem for dependent events*.* If A and B are dependent events, then:

$$P(A\cdot B)=P(A\,and\;B)=P(A)\cdot P(B|A)=P(B)\cdot P(A|B), \qquad (12.4)$$

where $P(B|A)$ or $P_A(B)$ is called the ***conditional probability*** of the event B given the event A (it means the probability that the event B will occur given that the event A has already occurred) and $P(A|B)$ is the conditional probability of the event A given the event B.

Example 12.4. There are 3 nonstandard electric bulbs among 50 electric ones. What is the probability that 2 electric bulbs taken at a time are nonstandard?

Solution. The probability of the event A that the first bulb is nonstandard equals $\frac{3}{50}$. The probability that the second bulb is nonstandard (the event B) on conditions that the first bulb is nonstandard (the event A) equals $\frac{2}{49}$, because the total number of bulbs and the number of nonstandard bulbs decreased by 1.

According to formula (12.4) we have

$$P(A\cdot B)=P(A)\cdot P(B|A)=\frac{3}{50}\cdot\frac{2}{49}\approx 0.0024.$$

Two random events A and B are said to be ***independent*** if the conditional probability of A given B coincides with the unconditional probability of A, i.e. $P(A|B)=P(A)$.

A ***conditional probability*** from formula (12.4) is expressed as:

$$P(B|A)=\frac{P(A\cdot B)}{P(A)}. \tag{12.5}$$

Example 12.5. The probability that it is Friday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Friday is 0.2. What is the probability that a student is absent given that today is Friday?

Solution. Let's denote that it is Friday as the event A and a student is absent as the event B. Then the event that a student is absent given that today is Friday is denoted by $B|A$. Let's find $P(B|A)$ using formula (12.5):

$$P(B|A)=\frac{P(A\cdot B)}{P(A)}=\frac{0.03}{0.2}=0.15.$$

A complete group of events

Events $A_1, A_2, \ldots, A_n$ are called ***pairwise independent*** if every possible pair of these events is independent, i.e. $P(A_i \cap A_j)=P(A_i)\cdot P(A_j)$ for any $i,\ j\ (i \neq j)$.

One says that events $A_1, A_2, \ldots, A_n$ form ***a complete group*** of pairwise incompatible events (or mutually exclusive), if exactly one of them necessarily occurs for each realization of the experiment and no other event can occur.

If events $A_1, A_2, \ldots, A_n$ form a complete group of pairwise incompatible events, then

$$P(A_1)+P(A_2)+\ldots+P(A_n)=1.$$

For example, two opposite events A and $\overline{A}$ form a complete group of incompatible events.

Example 12.6. Let the probability that the shooter scores 10 points, when hitting the target, equal 0.4, 9 points – 0.2, 8 points – 0.2, 7 points – 0.1, 6 points and fewer – 0.1. What is the probability that the shooter scores no less than 9 points with one shot?

Solution. Let A_1 be the shooter scoring 10 points, A_2 be the shooter scoring 9 points, A_3 be the shooter scoring 8 points, A_4 be the shooter scoring 7 points, A_5 be the shooter scoring 6 points and fewer.

These events form a complete group of pairwise incompatible events, i.e.

$$P(A_1)+P(A_2)+P(A_3)+P(A_4)+P(A_5)=1.$$

Let C be the shooter scoring no less than 9 points with one short.

The required event will occur (mark it C) if the shooter scores either 9 (the event A_2) or 10 points (the event A_1). The events A_2 and A_1 are incompatible. Thus, $P(C)=P(A_1)+P(A_2)=0.2+0.4=0.6$.

The notion of pairwise independence of random events. Independence in a totality

A pairwise independent collection of events $A_1, A_2, \ldots, A_n$ is called a set of events any two of which are independent.

Any collection of mutually independent events is *pairwise independent*.

Let events $A_1, A_2, \ldots, A_n$ be independent, A is "at least one of n events occurs in the experiment". Then $\overline{A}$ is this event that none of n events occurs in the experiment, i.e. $\overline{A}=\overline{A_1}\cdot\overline{A_2}\cdot\ldots\cdot\overline{A_n}$. The events A and $\overline{A}$ form a complete group of incompatible events, therefore,

$$P(A)=1-P(\overline{A})=1-P(\overline{A_1})\cdot P(\overline{A_2})\cdot\ldots P(\overline{A_n}). \qquad (12.6)$$

This formula (12.6) is called the ***probability that at least one of*** n ***events occurs***.

Let's denote $P(A_1)=p_1$, $P(\overline{A_1})=1-p_1=q_1$, $\ldots$, $P(A_n)=p_n$, $P(\overline{A_n})=1-p_n=q_n$ and transform formula (12.6):

$$P(A)=1-P(\overline{A_1})\cdot P(\overline{A_2})\cdot\ldots P(\overline{A_n})=1-q_1\cdot q_2\cdot\ldots\cdot q_n. \qquad (12.7)$$

Example 12.7. Three students are going to take an exam. The probability that the first student passes it equals 0.9, the second one is 0.75, the third one is 0.6. What is the probability that at least one of three students passes the exam?

Solution. Let A_1 be the event that the first student passes the exam, A_2 be the event that the second student passes the exam, A_3 be the event that the third one does it.

Each student can either pass the exam or not. Then $P(A_1)=p_1=0.9$, $P(\overline{A_1})=1-p_1=q_1=0.1$, $P(A_2)=p_2=0.75$, $P(\overline{A_2})=1-p_2=q_2=0.25$, $P(A_3)=p_3=0.6$, $P(\overline{A_3})=1-p_3=q_3=0.4$.

Events A_1, A_2, A_3 are independent. If the event A is such that at least one of three students passes the exam, then the complementary event $\overline{A}$ (not A) is such that no student passes the exam (it means $\overline{A_1}\cdot\overline{A_2}\cdot\overline{A_3}$).

Let's use formula (12.7) and obtain:

$$P(A)=1-q_1\cdot q_2\cdot q_3=1-0.1\cdot 0.25\cdot 0.4=1-0.01=0.99.$$

If all events $A_1, A_2, \ldots, A_n$ have equal probability, i.e. $P(A_1)=P(A_2)=$ $=\ldots=P(A_n)=p$, then $P(\overline{A_1})=P(\overline{A_2})=\ldots=P(\overline{A_n})=1-p=q$ and from formula (12.7) we have:

$$P(A)=1-q^n. \qquad (12.8)$$

Let's define the necessary number of trials (n) with the given reliability P no less than $P(A)$, i.e. $P(A)\geq P$, using formula (12.8):

$$P(A) = 1 - q^n \geq P$$

or

$$1 - (1 - p)^n \geq P$$

or

$$(1 - p)^n \leq 1 - P.$$

Let's take a natural logarithm of both parts of this inequality:

$$n \cdot \ln(1 - p) \leq \ln(1 - P).$$

Hence

$$n \leq \frac{\ln(1 - P)}{\ln(1 - p)}.$$

The total probability formula

Let's suppose that a complete group of pairwise incompatible events $H_1, H_2, \ldots, H_n$ is given and the unconditional probabilities $P(H_1), P(H_2), \ldots, P(H_n)$, as well as the conditional probabilities $P(A|H_1), P(A|H_2), \ldots, P(A|H_n)$ of an event A, are known. Then the probability of A can be determined by the ***total probability formula***

$$P(A) = \sum_{k=1}^{n} P(H_i) \cdot P(A|H_i). \tag{12.9}$$

Each of the events $H_1, H_2, \ldots, H_n$ is called ***hypothesis***.

$P(H_i)$ is called ***a priori probability*** (***premature probability***).

Example 12.8. Three machines produce the same type of product in a factory. The first one makes 200 articles, the second one 300 articles and the third one 500 articles. It is known that the first machine produces 1 % of defective articles, the second one 2 %, the third one 4 %. What is the probability that an article selected randomly from the total products will be defective?

Solution. Let A be the event that the chosen article is defective.

Let's consider the following complete group of events (hypotheses): H_1 denotes the event that the randomly selected article is made by the first machine, H_2 denotes the event that the randomly selected article is made by the second machine, H_3 denotes the event that the randomly selected article is made by the third machine.

Let's find their probabilities:

$$P(H_1) = \frac{200}{200+300+500} = \frac{200}{1000} = 0.2,$$

$$P(H_2) = \frac{300}{1000} = 0.3, \qquad P(H_3) = \frac{500}{1000} = 0.5.$$

Since events H_1, H_2 and H_3 form the complete group, then

$$P(H_1) + P(H_2) + P(H_3) = 0.2 + 0.3 + 0.5 = 1.$$

Let's define conditional probabilities $P(A|H_1), P(A|H_2), P(A|H_3)$:

$$P(A|H_1) = 0.01,\ P(A|H_2) = 0.02,\ P(A|H_3) = 0.04.$$

Here $A|H_1$ is a defective article produced by the first machine, $A|H_2$ is a defective article produced by the second machine, $A|H_3$ is a defective article produced by the third machine.

Let's use the total probability formula (12.9) and find:

$$P(A) = P(H_1)\cdot P(A|H_1) + P(H_2)\cdot P(A|H_2) + P(H_3)\cdot P(A|H_3) =$$
$$= 0.2\cdot 0.01 + 0.3\cdot 0.02 + 0.5\cdot 0.04 = 0.028.$$

We have 2.8 % of defective articles in the total production.

Bayes' formula

If it is known that the event A has occurred but it is unknown which of the events $H_1, H_2, \ldots, H_n$ has occurred, then ***Bayes' formula*** is used:

$$P(H_k|A) = \frac{P(H_k)\cdot P(A|H_k)}{P(A)},\ k = 1,2,\ldots,n \qquad (12.10)$$

$$\text{and } P(H_1|A) + P(H_2|A) + \ldots + P(H_n|A) = 1.$$

where $P(H_i|A)$ is called ***a posteriori probability*** *(final probability)*.

Example 12.9. Let's use the condition of example 6 and solve the following problem. It is known that a selected article is defective. What is the probability that this article was made by the second machine?

Solution. The desired probability of the event $H_2|A$ (the selected article was made by the second machine provided it is known that it is defective) is determined by Bayes' formula (12.10):

$$P(H_2|A) = \frac{P(H_2)\cdot P(A|H_2)}{P(A)} = \frac{0.3\cdot 0.02}{0.028} = \frac{6}{28} = \frac{3}{14}.$$

Theme 13. The scheme of independent trials

Bernoulli's formula

Trials in which events occurring in distinct trials are independent are said to be ***independent****.* Here the probability of each event A of the form $A = A_1 \cdot A_2 \cdot \ldots \cdot A_n$ is defined as $P(A) = P(A_1)\cdot P(A_2)\cdot \ldots \cdot P(A_n)$*.*

Let independent events occur in n independent trials. In each trial the event A can occur or can't occur.

A sequence of n independent trials is also called a ***Bernoulli scheme****.*

In this case, some event A occurs with probability $p = P(A)$ (the probability of "success") and does not occur with probability $q = P(\overline{A}) = 1 - P(A) = 1 - p$ (the probability of "failure") in each trial.

If k is the number of occurrences of the event A (the number of "successes") in n independent Bernoulli trials, then the probability that A occurs exactly k times is given by the formula:

$$P(k) = P_n(k) = C_n^k p^k q^{n-k}, \qquad (13.1)$$

where $C_n^k = \dfrac{n!}{k! \cdot (n-k)!}$ is a combination of n things taken k at a time.

This relation is called the ***Bernoulli formula*** (***binomial distribution***).

The probability that the event occurs at least m times in n independent trials is calculated by the formula:

$$P_n(k \geq m) = \sum_{k=m}^{n} C_n^k p^k q^{n-k} = 1 - \sum_{k=0}^{m-1} C_n^k p^k q^{n-k}.$$

The probability that the event occurs at least once in n independent trials is calculated by the formula:

$$P_n(k \geq 1) = 1 - q^n.$$

The probability that the event A occurs no less than k_1 and no more than k_2 times ($k_1 < k_2$) satisfies the relation:

$$P_n(k_1 \leq k \leq k_2) = P_n(k_1) + P_n(k_1 + 1) + \ldots + P_n(k_2) =$$
$$= C_n^{k_1} p^{k_1} q^{n-k_1} + \ldots + C_n^{k_2} p^{k_2} q^{n-k_2}.$$

Example 13.1. The probability of a train's arrival at a station on time is equal to 0.8. What is the probability that out of 4 expected trains 2 trains will arrive on time?

Solution. Let A be a train arriving at a station on time, $P(A) = p = 0.8$. Then $\overline{A}$ is a train that doesn't arrive at a station on time and

$$q = P(\overline{A}) = 1 - P(A) = 1 - 0.8 = 0.2.$$

Here $n = 4 < 30$, $k = 2$.

According to the Bernoulli formula (13.1) we have

$$P_4(2)=C_4^2\cdot 0.8^2\cdot 0.2^{4-2}=\frac{4!}{2!\cdot(4-2)!}\cdot 0.8^2\cdot 0.2^2=6\cdot 0.64\cdot 0.04=0.1536.$$

The most probable number of successes and its probability

The number k_0 of occurrences of the event A in the independent trials is called the ***most probable number*** if the probability of the event occurring such number k_0 times is maximum (the largest value).

Let the event A occur with probability $p=P(A)$ and do not occur with probability $q=P(\overline{A})=1-P(A)=1-p$ in the trial. Then the ***most probable number*** k_0 is defined by the inequality:

$$np-q\le k_0\le np+p, \qquad (13.2)$$

where k_0 is an integer.

Example 13.2. The probability of finding a mistake on a book page is equal to 0.002. 500 pages are checked. Find the most probable number of pages with mistakes.

Solution. Let A be finding a mistake on a book page, $P(A)=p=0.002$. Then $\overline{A}$ is lack of a mistake on a book page and

$$q=P(\overline{A})=1-P(A)=1-0.002=0.998.$$

According to (13.2) we have

$$500\cdot 0.002-0.998\le k_0\le 500\cdot 0.002+0.002$$

or

$$1-0.998\le k_0\le 1+0.002$$

or

$$0.002\le k_0\le 1.002.$$

Then $k_0=1$.

It is very difficult to use Bernoulli's formula for large n and k. In this case, one has to use approximate formulas for calculating $P_n(k)$ with desired accuracy.

The local theorem of de Moivre – Laplace

Suppose that the number of independent trials increases unboundedly ($n \to \infty$ or n approaches infinity) and the probability $p = const$, $0 < p < 1$, then the probability $P_n(k)$ that A occurs exactly k times out of n satisfies the limit relation

$$P_n(k) \approx \frac{1}{\sqrt{npq}} \varphi\left(\frac{k-np}{\sqrt{npq}}\right), \tag{13.3}$$

where the limit expression $\varphi(x)$ is Laplace differential function or the probability density of the standard normal distribution, i.e.

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

This function is even, i.e. $\varphi(-x) = \varphi(x)$.

The function value ($x < 4$) is defined by the Laplace differential function table (appendix A). For the values $x > 4$ $\varphi(x) \approx 0$.

Example 13.3. The probability of the birth of a boy is equal to 0.51. Find the probability that among 200 newborns there will be the same number of boys and girls.

Solution. Let A be the birth of a boy, $P(A) = p = 0.51$. Then $\overline{A}$ is the birth of a girl and $P(\overline{A}) = 1 - P(A) = 1 - 0.51 = 0.49$. Here $n = 200$, $k = 100$.

According to the local theorem of de Moivre – Laplace (13.3) we have

$$P_{200}(100) \approx \frac{1}{\sqrt{200 \cdot 0.51 \cdot 0.49}} \varphi\left(\frac{100 - 200 \cdot 0.51}{\sqrt{200 \cdot 0.51 \cdot 0.49}}\right) = \frac{\varphi(-0.28)}{7.07}.$$

The function $\varphi(x)$ is even, then we obtain $\varphi(-0.28)=\varphi(0.28)$. Let's apply the Laplace differential function table (appendix A) and have $\varphi(0.28)=0.3836$. Let's substitute this value into the previous formula and obtain:

$$P_{200}(100) \approx \frac{\varphi(-0.28)}{7.07} = \frac{\varphi(0.28)}{7.07} = \frac{0.3836}{7.07} \approx 0.0543.$$

The integral theorem of de Moivre – Laplace

Let's suppose that $n \to \infty$ and the probability $p = const$, $0 < p < 1$, then the probability $P_n(k)$ that A occurs no less than k_1 and no more than k_2 times ($k_1 < k_2$) satisfies the limit relation:

$$P_n(k_1 \le k \le k_2) \approx \Phi\left(\frac{k_2 - np}{\sqrt{npq}}\right) - \Phi\left(\frac{k_1 - np}{\sqrt{npq}}\right), \qquad (13.4)$$

where the limit expression $\Phi(x)$ is the Laplace integral function or the cumulative distribution function of the standard normal distribution, i.e.:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{x^2}{2}}\, dx.$$

This function is odd, i.e. $\Phi(-x) = -\Phi(x)$.

The function value ($x < 4$) is defined by the Laplace integral function table (appendix B). For values $x > 4$ $\Phi(x) \approx 0.5$.

Example 13.4. The probability of the birth of a girl is equal to 0.49. Find the probability that among 200 newborns there will be

a) from 95 to 110 girls;

b) no less 117 girls;

c) no more 120 girls.

Solution. Let A be the birth of a girl, $P(A) = p = 0.49$. Then $\overline{A}$ is the birth of a boy and $q = P(\overline{A}) = 1 - P(A) = 1 - 0.49 = 0.51$.

a) Here $n = 200$, $k_1 = 95$, $k_2 = 110$.

According to the integral theorem of de Moivre – Laplace (13.4) we have

$$P_{200}(95 \le k \le 110) \approx \Phi\left(\frac{110 - 200 \cdot 0.49}{\sqrt{200 \cdot 0.49 \cdot 0.51}}\right) - \Phi\left(\frac{95 - 200 \cdot 0.49}{\sqrt{200 \cdot 0.49 \cdot 0.51}}\right) =$$

$$= \Phi\left(\frac{12}{7.07}\right) - \Phi\left(\frac{-3}{7.07}\right) = \Phi(1.70) - \Phi(-0.42).$$

The function $\Phi(x)$ is odd, then $\Phi(-0.42) = -\Phi(0.42)$. Let's apply the Laplace integral function table (appendix B) and have $\Phi(1.70) = 0.4554$ and $\Phi(0.42) = 0.1628$. Let's substitute these values into the previous formula and obtain:

$$P_{200}(95 \le k \le 110) \approx 0.4554 + 0.1628 = 0.6182.$$

b) The condition "no less 117 girls" means from 117 to 200 girls.

Here $n = 200$, $k_1 = 117$, $k_2 = 200$.

According to the integral theorem of de Moivre – Laplace (13.4) we have

$$P_{200}(117 \le k \le 200) \approx \Phi\left(\frac{200 - 200 \cdot 0.49}{\sqrt{200 \cdot 0.49 \cdot 0.51}}\right) - \Phi\left(\frac{117 - 200 \cdot 0.49}{\sqrt{200 \cdot 0.49 \cdot 0.51}}\right) =$$

$$= \Phi\left(\frac{102}{7.07}\right) - \Phi\left(\frac{19}{7.07}\right) = \Phi(14.43) - \Phi(2.69).$$

Let's use the property of the function $\Phi(x)$ and get $\Phi(x) \approx 0.5$ for values $14.43 > 4$. Then let's apply the Laplace integral function table (appendix B) and have $\Phi(2.69) = 0.4964$.

Let's substitute these values into formula (13.4) and obtain:

$$P_{200}(117 \le k \le 200) \approx \Phi(14.43) - \Phi(2.69) = 0.5 - 0.4964 = 0.0036.$$

c) The condition "no more 120 girls" means from 0 to 120 girls. Here $n = 200$, $k_1 = 0$, $k_2 = 120$. Let's use formula (13.4) and obtain:

$$P_{200}(0 \le k \le 120) \approx \Phi\left(\frac{120 - 200 \cdot 0.49}{\sqrt{200 \cdot 0.49 \cdot 0.51}}\right) - \Phi\left(\frac{0 - 200 \cdot 0.49}{\sqrt{200 \cdot 0.49 \cdot 0.51}}\right) =$$
$$= \Phi\left(\frac{22}{7.07}\right) - \Phi\left(-\frac{98}{7.07}\right) = \Phi(3.11) - \Phi(-13.86) =$$
$$= \Phi(3.11) + \Phi(13.86) = 0.4990 + 0.5 = 0.999.$$

The Poisson theorem

If the number of independent trials increases unboundedly ($n \to \infty$) and the probability p simultaneously decays ($p \to 0$) so that their product np is a constant ($np = \lambda = const$), then the probability $P_n(k)$ satisfies the limit relation:

$$P_n(k) \approx \frac{\lambda^k}{k!} e^{-\lambda}. \tag{13.5}$$

The probability that the event A occurs no less than k_1 and no more than k_2 times ($k_1 < k_2$) satisfies the relation:

$$P_n(k_1 \le k \le k_2) = P_n(k_1) + P_n(k_1 + 1) + ... + P_n(k_2) = \frac{\lambda^{k_1}}{k_1!} e^{-\lambda} + ... + \frac{\lambda^{k_2}}{k_2!} e^{-\lambda}.$$

Example 13.5. The probability of finding a mistake on a book page is equal to 0.002. 1000 pages are checked. Find the probability that there is a mistake on 3 pages.

Solution. Let A be finding a mistake on a book page, $P(A) = p = 0.002$.

Then $\overline{A}$ is lack of a mistake on a book page and

$$q = P(\overline{A}) = 1 - P(A) = 1 - 0.002 = 0.998.$$

Here $n = 1000$, $k = 3$. Then $\lambda = 1000 \cdot 0.002 = 2$.
According to the Poisson formula (13.5) we have:

$$P_{1000}(3) \approx \frac{\lambda^3}{3!} e^{-\lambda} = \frac{2^3}{3!} e^{-2} = \frac{8}{6} \cdot (2.72)^{-2} \approx 0.1802.$$

The probability of deviation of relative frequency from the probability

Let some event A occur with probability $p = P(A)$, $0 < p < 1$ and don't occur with probability $q = P(\overline{A}) = 1 - P(A) = 1 - p$ in each of n independent trials.

It is necessary to define the probability of deviation of relative frequency from the constant probability, i.e. find the probability of inequality $\left|\frac{m}{n} - p\right| \le \varepsilon$. Then the probability of an absolute value of deviation of relative frequency from its constant probability less than or equal to ε equals $2\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right)$, i.e.:

$$P\left(\left|\frac{m}{n} - p\right| \le \varepsilon\right) \approx 2\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right),$$

where $\frac{m}{n}$ is a relative frequency, p is the constant probability of A, n is the number of trials, ε is an accuracy; $\Phi(x)$ is the Laplace integral function (appendix B).

Example 13.6. For defining the level of students' knowledge in the given subject 100 students are given tests. The probability of carrying out a test excellently is 0.1. Find:

a) the probability P that the relative frequency deviates from the probability p by the value $\varepsilon = 0.01$;

b) the accuracy ε, at which the probability of deviation of relative frequency from the probability p is $P = 0.95$;

c) how many students it is necessary to take that with the accuracy $\varepsilon = 0.02$ for the probability of deviation of relative frequency from the probability p to be $P = 0.9$.

Solution. a) Let's find $P\left(\left|\frac{m}{n} - p\right| \le \varepsilon\right)$. If $p = 0.1$, then

$$q = 1 - p = 1 - 0.1 = 0.9.$$

Let's substitute:

$$P\left(\left|\frac{m}{n} - 0.1\right| \le 0.01\right) \approx 2\Phi\left(0.01\sqrt{\frac{100}{0.1 \cdot 0.9}}\right) = 2\Phi(0.33) = 2 \cdot 0.1293 = 0.2586.$$

b) According to the condition $P\left(\left|\frac{m}{n} - p\right| \le \varepsilon\right) = 0.95$. Let's find ε. Then

$$2\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right) = 0.95;$$

$$\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right) = 0.95/2$$

or

$$\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right) = 0.475.$$

Using appendix B we get $\Phi(1.96) = 0.475$. Thus,

$$\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right) = \Phi(1.96)$$

or

$$\varepsilon\sqrt{\frac{n}{pq}} = 1.96.$$

Let's express ε and get:

$$\varepsilon = 1.96\sqrt{\frac{pq}{n}}.$$

Let's substitute:

$$\varepsilon = 1.96\sqrt{\frac{pq}{n}} = 1.96\sqrt{\frac{0.1 \cdot 0.9}{100}} = 0.0588 \approx 0.06.$$

c) According to the condition $P\left(\left|\frac{m}{n} - p\right| \le \varepsilon\right) = 0.9$ and $\varepsilon = 0.02$. Let's find n. Then

$$P\left(\left|\frac{m}{n} - p\right| \le \varepsilon\right) = 0.9 = 2\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right).$$

Thus $\varepsilon\sqrt{\frac{n}{pq}} = 1.65$.

Let's substitute:

$$0.02\sqrt{\frac{n}{0.1 \cdot 0.9}} = 1.65$$

or

$$n = \frac{1.65^2 \cdot 0.1 \cdot 0.9}{0.02^2} \approx 613.$$

Thus, it is necessary to take 613 students.

Tasks for independent work

Theme 11. Empirical and logical bases of probability theory
Theme 12. Basic theorems of probability theory, their economic meaning

Task 1. 20 students are taking part in a chess tournament. How many games will they play if one game has to be played by any 2 players?

Task 2. How many different ways are there to choose a chairman, his deputy, and a first and second assistant for them out of 30 participants at an election meeting?

Task 3. Find the number of possible ways of distributing 3 tickets among 10 students.

Task 4. If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary,

a) how many variants are there to select the dictionary and 2 novels?

b) how many variants are there to select 1 novel and 2 books of poems?

Task 5. How many three-digit numbers can be formed from the digits 1, 2, 3, 4, 5, if each digit can be used only once (with repetitions)?

Task 6. How many four-digit numbers can be formed from the digits 0, 1, 3, 4, 6, if each digit can be used only once (with repetitions)?

Task 7. How many different rearrangements of the letters in the word

(a) EDUCATION, (b) HONORS, (c) MISSISSIPPI are there?

Task 8. In a classroom there are 3 pupils and 3 chairs standing in a row. In how many different orders can the pupils sit on these chairs?

Task 9. How many different possibilities are there for any 2 of 3 pupils to sit on 2 chairs?

Task 10. How many three-letter words can we make with the letters in the word LOVE?

Task 11. There are 6! permutations of the 6 letters of the word *square*.

a) In how many of them is *r* the second letter? _ r _ _ _ _

b) In how many of them are *q* and *e* next to each other?

Task 12. (a) How many different arrangements are there of the letters of the word *numbers*? (b) How many of those arrangements have *b* as the first letter? (c) How many have *b* as the last letter -- or in any specified position? (d) How many will have *n*, *u*, and *m* together?

Task 13. (a) How many different arrangements (permutations) are there of the digits 0, 1, 2, 3, 4? (b) How many 5-digit numbers can you make of those digits, in which the first digit is not 0, and no digit is repeated? (c) How many 5-digit odd numbers can you make, and no digit is repeated?

Task 14. In how many ways can you select a committee of 3 students out of 10 students?

Task 15. A committee including 3 boys and 4 girls is to be formed from a group of 10 boys and 12 girls. How many different committees can be formed from the group?

Task 16. In a certain country, the car number plate is formed by 4 digits out of the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 followed by 3 letters from the alphabet. How many number plates can be formed?

Task 17. One digit is randomly chosen from 1 to 9. What is the probability that the chosen number is: 1) even; 2) odd; 3) prime; 4) greater than 7; 5) less than 4; 6) composite; 7) even or prime; 8) odd or prime; 9) composite or even; 10) composite or odd; 11) composite or prime; 12) composite and prime?

Task 18. A symmetrical dice is rolled four times. What is the probability of getting the faces with the odd, even, odd, even points respectively?

Task 19. A student is choosing 3 questions out of 30 at an exam. There are 10 questions in algebra, 15 in analysis and 5 in geometry. What is the probability that he will choose at least two questions in the same area?

Task 20. A box of fuses which are all of the same shape and size comprises 23 2A fuses, 47 5A fuses and 69 13A fuses. Determine the probability of selecting 2 or more 5A fuses.

Task 21. A phone company has found that 75 % of customers want text messaging, 80 % photo capabilities and 65 % both. What is the probability that a customer will want at least one of these?

Task 22. On five cards, there are written letters "s", "m", "e", "t", "a". After shuffling they take five cards one by one and put them near in turn. What is the probability that the word results in: 1) "steam"; 2) "team"; 3) "tea"?

Task 23. There are N white and M black balls (N good and M defective articles on the shelf). If k balls are chosen randomly, what is the probability of getting this way exactly n white balls?

Task 24. Out of the set of 52 playing cards, 5 cards were chosen randomly. What is the probability of getting exactly two hearts among them?

Task 25. Three students are going to take an exam. The probability that the first student will pass it equals 0.6; for the second and third ones it is 0.7 and 0.75 respectively. What is the probability that

a) all of the three students will pass the exam;

b) the first student will only do it;

c) the second and the third ones will only do it;

d) one student will do it;

e) no student will do it;

f) at least one student will do it;

g) at least two students will do it?

Task 26. Three machines produce the same type of product at a factory. The first one gives 200 articles, the second one makes 300 articles and the third one produces 500 articles. It is known that the first machine produces 1 % of defective articles, the second one does 2 %, the third one does 4 %.

a) What is the probability that the article selected randomly from the total products will be defective?

b) It is known that the selected article is defective. What is the probability that this article was made by the first machine?

c) It is known that the selected article is defective. What is the probability that this article was made by the second machine?

d) What is the probability that the article selected randomly from the total products will be standard?

e) It is known that the selected article is standard. What is the probability that this article was made by the first machine?

f) It is known that the selected article is standard. What is the probability that this article was made by the third machine?

Task 27. Suppose the probability that a married man votes is 0.45, the probability that a married woman votes is 0.4, and the probability a woman votes given that her husband votes is 0.6. What is the probability that: a) both vote; b) a man votes given that his wife votes?

Task 28. Three girls, Alice, Betty and Charlotte, wash the family dishes. Since Alice is the eldest, she does the job 40 % of the time. Betty and Charlotte share the other 60 % equally. The probability that at least one dish will be broken when Alice is washing them is 0.02; for Betty and Charlotte the probabilities are 0.03 and 0.02. The parents do not know who is washing the dishes, but one night they hear one break. What is the probability that Alice was washing up? Betty? Charlotte?

Task 29. Three machines produce the same type of product at a factory. The first one produces 150 articles, the second one makes 600 articles and the third machine manufactures 250 articles. It is known that the share of defective articles is 2 % with the first machine, 4 % with the second one, 8 % with the third one.

a) What is the probability that an article selected randomly from the total products will be defective?

b) It is known that a selected article is defective. What is the probability that this article was made by the third machine?

c) What is the probability that an article selected randomly from the total products will be fully functioning?

d) It is known that a selected article is fully functioning. What is the probability that this article was made by the second machine?

Theme 13. The scheme of independent trials

Task 1. The probability of the birth of a boy is equal to 0.51. Make up the distribution law of a number of newborn boys out of 6 newborns. Find the probability that among 10 newborns there will be from 3 to 5 boys.

Task 2. The probability that the student will pass a test at the first attempt is equal to 0.9. Find the probability that out of 7 students of the same knowledge level:

a) 5 students will pass the test; b) from 4 to 6 will pass the test.

c) Find the most probable number of students who will pass the test at the first attempt.

Task 3. The probability of the train arrival at the station on time is equal to 0.8. What is the probability that out of 4 expected trains

a) 3 trains will arrive on time;

b) no less than 3 will arrive on time;

c) no more than 3 will arrive on time;

d) at least one will arrive on time?

e) Find the most probable number of the train arrival at the station on time.

f) Find the most probable number of the train nonarrival at the station on time and the probability of the most probable number.

Task 4. The probability that the part is produced with a defect equals 0.2. Find the probability that among 400 randomly selected parts there will be:

a) from 70 to 100 defective ones; b) 90 defective ones;

c) no less than 60 defective ones; d) no more than 100 defective ones.

e) Find the most probable number of defective articles.

f) Find the most probable number of standard articles.

Task 5. The probability of hitting the target with 1 shot equals 0.6. Find the probability of the following events:

a) with 12 shots the target will be hit 7 times;

b) with 15 shots the target will be hit from 5 to 8 times;

c) with 200 shots the target will be hit

1) 120 times;

2) from 90 to 110 times;

3) no less than 111 and no more than 130 times;

4) no more than 110 times;

5) no less than 115 times.

d) Find the most probable number of hitting the target with 1 shot out of 12 shots and the probability of the most probable number.

e) Find the most probable number of hitting the target with 1 shot out of 200 shots and the probability of the most probable number.

Task 6. A factory sent 5000 products of high quality to the warehouse. The probability of damaging the products on the way is equal to 0.0008. Find the probability that:

a) 5 damaged products will be received at the warehouse;

b) from 3 to 6 damaged products will be received at the warehouse;

c) at least one damaged product will be received at the warehouse.

Task 7. The probability of hitting the target with 1 shot equals 0.6. Find the probability of the following events:

a) with 600 shots the hitting frequency will deviate from the probability 0.6 by the absolute value of no more than 0.03;

b) find the boundaries of hitting the target with 600 shots in order that the probability $P = 0.993$;

c) find such a number of shots that the probability $P = 0.993$ gives the deviation of frequency of hits from the probability 0.6 by the absolute value no more 0.03;

d) the accuracy ε, at which the probability of deviation of relative frequency from the probability $p = 0.7$ is $P = 0.996$.

Individual tasks

Variants 1 – 30

Using Table 1 solve the following tasks:

Task 1. A student knows m out of n questions of a discipline's program. A lecturer asks three questions.

What is the probability that the student will answer

(a) only one question;

(b) only two questions;

(c) all of the three questions;

(d) at least one question;

(e) at least two questions?

(f) What is the probability that a student won't answer any question?

Task 2. Three students are going to take an exam. The probability that the first student will pass it equals p_1; for the second and third ones it is p_2 and p_3 respectively.

What is the probability that

(a) all of the three students will pass the exam;

(b) the first student will only do it;

(c) the second and the third ones will only do it;

(d) one student will do it;

(e) no student will do it;

(f) at least one student will do it?

Task 3. In a factory, machines A, B and C are all producing metal rods of the same length. Machine A produces P_1 % of the rods, machine B produces P_2 % and the rest are produced by machine C. Out of their production of rods, machines A, B and C produce Q_1 %, Q_2 % and Q_3 % defective rods respectively.

(a) Find the probability that a randomly selected rod is defective.

(b) Given that a randomly selected rod is defective, find the probability that it was produced by machine A.

(c) Given that a randomly selected rod is defective, find the probability that it was produced by machine C.

(d) Find the probability that a randomly selected rod is fully functioning.

(e) Given that a randomly selected rod is fully functioning, find the probability that it was produced by machine B.

Table 1

Data for individual tasks

Variants	Task 1		Task 2			Task 3				
	m	n	p_1	p_2	p_3	P_1 %	P_2 %	Q_1 %	Q_2 %	Q_3 %
1	20	80	0.1	0.3	0.6	15	50	1	3	5
2	12	60	0.2	0.4	0.7	35	40	2	5	6
3	15	75	0.3	0.6	0.8	20	65	3	4	2
4	11	55	0.4	0.7	0.9	30	55	4	6	1
5	19	76	0.6	0.1	0.3	45	15	5	7	3
6	13	65	0.7	0.2	0.4	60	25	3	5	1
7	18	54	0.8	0.3	0.6	50	35	5	6	2
8	17	51	0.9	0.4	0.7	65	15	4	2	3
9	11	88	0.1	0.2	0.3	20	45	6	1	4
10	15	90	0.2	0.3	0.4	45	30	7	3	5
11	19	57	0.3	0.4	0.6	50	15	5	1	3
12	18	72	0.4	0.6	0.8	40	35	6	2	5
13	11	66	0.2	0.3	0.1	65	20	2	3	4
14	16	64	0.3	0.4	0.2	55	30	1	4	6
15	18	90	0.4	0.6	0.3	15	45	3	5	7
16	15	60	0.6	0.8	0.4	25	60	2	3	4
17	14	70	0.7	0.8	0.9	35	50	4	5	7
18	12	96	0.4	0.9	0.6	15	65	9	6	1
19	17	68	0.6	0.7	0.8	45	20	7	4	2
20	11	77	0.3	0.8	0.9	30	45	5	2	3
21	12	72	0.9	0.7	0.8	10	55	4	2	3
22	14	84	0.6	0.4	0.9	60	15	7	4	5
23	10	80	0.8	0.6	0.7	45	25	1	9	6
24	13	78	0.9	0.3	0.8	50	20	2	7	4
25	15	90	0.4	0.6	0.8	70	10	3	5	2
26	16	80	0.6	0.3	0.1	30	60	3	4	2
27	20	60	0.7	0.4	0.2	40	25	5	7	4
28	10	70	0.8	0.6	0.3	15	50	6	1	9
29	13	52	0.9	0.7	0.4	45	35	4	2	7
30	14	56	0.1	0.8	0.3	50	30	2	3	5

Using Table 2 solve the following tasks:

Task 4. The probability of hitting the target with 1 shot equals p.

Find the probability of the following events:

a) with n shots the target will be hit

1) k_1 times;
2) at least 1 time;
3) no less than k_1 and no more than k_2 times;
4) no more than k_1 times;
5) no less than k_2.

b) Find the most probable number of hitting the target with 1 shot out of n shots.

c) Find the probability of the most probable number.

Task 5. The pizza delivery department receives p brand pizza orders.

Find the probability that among n received orders there will be

a) k_1 brand pizza orders;

b) k_2 brand pizza orders;

c) no less than k_1 brand pizza orders;

d) no more than k_2 brand pizza orders;

e) from k_3 to k_4 brand pizza orders;

f) from k_4 to k_5 brand pizza orders.

g) Find the most probable number of brand pizza orders.

h) Find the probability of the most probable number.

Using Table 3 solve the following task:

Task 6. The factory sent n products of high quality to the warehouse. The probability of damaging the products on the way is equal to p.

Find the probability that

a) k_1 damaged products will be received at the warehouse;

b) from k_2 to k_3 damaged products will be received at the warehouse;

c) at least one damaged product will be received at the warehouse.

d) Find the most probable number of damaged products.

e) Find the probability of the most probable number.

Table 2

Data for individual tasks

Variants	Task 4				Task 5						
	n	p	k_1	k_2	n	p	k_1	k_2	k_3	k_4	k_5
1	5	0.1	2	4	100	0.9	92	87	85	97	120
2	6	0.2	3	5	200	0.8	156	163	154	165	190
3	7	0.3	4	6	300	0.7	213	206	204	215	240
4	8	0.4	5	7	400	0.6	242	236	235	247	270
5	9	0.6	6	8	100	0.4	42	37	35	46	60
6	5	0.7	2	4	200	0.3	63	58	55	66	80
7	6	0.8	3	5	300	0.2	62	56	56	65	90
8	7	0.9	4	6	400	0.1	43	38	36	45	50
9	8	0.1	5	7	100	0.9	93	88	87	95	130
10	9	0.2	6	8	200	0.8	162	157	155	164	180
11	5	0.3	2	4	300	0.7	212	207	205	214	230
12	6	0.4	3	5	400	0.6	245	236	237	245	280
13	7	0.6	4	6	100	0.4	41	38	36	45	70
14	8	0.7	5	7	200	0.3	61	57	56	65	90
15	9	0.8	6	8	300	0.2	62	56	55	66	60
16	5	0.9	2	4	400	0.1	42	37	35	46	140
17	6	0.1	3	5	100	0.9	87	92	85	97	190
18	7	0.2	4	6	200	0.8	163	156	154	165	240
19	8	0.3	5	7	300	0.7	206	213	204	215	290
20	9	0.4	6	8	400	0.6	236	242	235	247	80
21	5	0.6	2	4	100	0.4	37	42	35	46	100
22	6	0.7	3	5	200	0.3	58	63	55	66	50
23	7	0.8	4	6	300	0.2	56	62	56	65	130
24	8	0.9	5	7	400	0.1	38	43	36	45	180
25	9	0.1	6	8	100	0.9	88	93	87	95	230
26	5	0.2	2	4	200	0.8	157	162	155	164	270
27	6	0.3	3	5	300	0.7	207	212	205	214	70
28	7	0.4	4	6	400	0.6	236	245	237	245	90
29	8	0.6	5	7	100	0.4	38	41	36	45	60
30	9	0.7	6	8	200	0.3	57	61	56	65	120

Using Table 3 solve the following task:

Task 7. The probability of hitting the target with 1 shot equals p.

a) Find the probability of the following event: with n shots the hitting frequency will deviate from the probability p by the absolute value no more than ε.

b) Find the probability of the following event: with n shots the hitting frequency will deviate from the probability p by the absolute value no more than ε.

c) Find the boundaries of hitting the target provided n shots and the given probability P_1.

d) Find such a number of shots that given the probability P_1 one can expect the deviation of frequency of hits from the probability p by the absolute value not going beyond ε.

e) Find the accuracy ε, at which the probability of deviation of relative frequency from the probability p is P_2.

Table 3

Data for individual tasks

Variants	Task 6					Task 7				
	n	p	k_1	k_2	k_3	n	p	ε	P_1	P_2
1	2	3	4	5	6	7	8	9	10	11
1	500	0.004	3	2	4	700	0.6	0.01	0.9624	0.9750
2	1000	0.002	3	3	5	300	0.8	0.02	0.9642	0.9850
3	2000	0.001	3	2	4	400	0.9	0.03	0.9660	0.9930
4	400	0.005	3	3	5	100	0.1	0.04	0.9676	0.9960
5	1000	0.003	4	3	5	200	0.2	0.05	0.9692	0.9970
6	2000	0.002	5	6	7	300	0.3	0.06	0.9708	0.9973
7	500	0.004	3	2	4	400	0.4	0.07	0.9722	0.9750
8	1000	0.004	5	4	6	300	0.7	0.08	0.9736	0.9850
9	2000	0.003	7	6	8	200	0.1	0.09	0.9762	0.9930
10	400	0.005	3	2	4	700	0.6	0.01	0.9774	0.9960
11	1000	0.005	6	5	7	300	0.8	0.02	0.9786	0.9970
12	3000	0.002	7	6	8	400	0.9	0.03	0.9796	0.9973
13	4000	0.001	5	4	6	100	0.1	0.04	0.9808	0.9750

Table 3 (the end)

1	2	3	4	5	6	7	8	9	10	11
14	500	0.004	4	3	5	200	0.2	0.05	0.9818	0.9850
15	1000	0.002	4	3	5	300	0.3	0.06	0.9826	0.9930
16	2000	0.001	4	3	5	400	0.4	0.07	0.9624	0.9960
17	400	0.005	4	3	5	300	0.7	0.08	0.9642	0.9970
18	1000	0.003	5	4	6	200	0.1	0.09	0.9660	0.9973
19	2000	0.002	6	5	7	700	0.6	0.01	0.9676	0.9750
20	500	0.004	4	3	5	300	0.8	0.02	0.9692	0.9850
21	1000	0.004	6	5	7	400	0.9	0.03	0.9708	0.9930
22	2000	0.003	8	7	9	100	0.1	0.04	0.9722	0.9960
23	400	0.005	4	3	5	200	0.2	0.05	0.9736	0.9970
24	1000	0.005	7	6	8	300	0.3	0.06	0.9762	0.9973
25	3000	0.002	8	7	9	400	0.4	0.07	0.9774	0.9750
26	4000	0.001	6	5	7	300	0.7	0.08	0.9786	0.9850
27	500	0.004	3	2	4	200	0.1	0.09	0.9796	0.9930
28	1000	0.002	3	3	5	700	0.6	0.01	0.9808	0.9960
29	2000	0.001	3	2	4	300	0.8	0.02	0.9818	0.9970
30	400	0.005	3	3	5	400	0.9	0.03	0.9826	0.9973

Theoretical questions

1. A stochastic experiment.
2. A random event.
3. A probabilistic space.
4. An outcome.
5. An impossible event.
6. A sure event.
7. Equally likely events.
8. Elementary events.
9. An intersection, a union, a difference of events.
10. The theorem of a sum of compatible events.
11. The theorem of a sum of incompatible events.
12. A classical definition of a probability.
13. A geometrical definition of a probability.
14. A statistical definition of a probability.
15. Permutations, arrangements, combinations with repetitions.
16. Permutations, arrangements, combinations without repetitions.
17. The rule of a sum.
18. The rule of a product.
19. The inclusion-exclusion principle.
20. A conditional probability.
21. The theorem of a product for dependent events.
22. The theorem of a product for independent events.
23. The notion of a pairwise independence of random events.
24. A complete group of events.
25. The formulas of a total probability and Bayes.
26. Repeated independent trials.
27. Bernoulli's scheme.
28. A binomial distribution.
29. The most probable number of successes and its probability.
30. The local theorem of Moivre – Laplace.
31. The integral theorem of Moivre – Laplace.
32. Poisson's theorem.

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Appendices

Appendix A

The values of the Laplace differential function $\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

x	0	1	2	3	4	5	6	7	8	9
0.0	0.3989	0.3989	0.3989	0.3988	0.3986	0.3984	0.3982	0.3980	0.3977	0.3973
0.1	0.3970	0.3965	0.3961	0.3956	0.3951	0.3945	0.3939	0.3932	0.3925	0.3918
0.2	0.3910	0.3902	0.3894	0.3885	0.3876	0.3867	0.3857	0.3847	0.3836	0.3825
0.3	0.3814	0.3802	0.3790	0.3778	0.3765	0.3752	0.3739	0.3726	0.3712	0.3697
0.4	0.3683	0.3668	0.3652	0.3637	0.3621	0.3605	0.3589	0.3572	0.3555	0.3538
0.5	0.3521	0.3503	0.3485	0.3467	0.3448	0.3429	0.3410	0.3391	0.3372	0.3352
0.6	0.3332	0.3312	0.3292	0.3271	0.3251	0.3230	0.3209	0.3187	0.3166	0.3144
0.7	0.3123	0.3101	0.3079	0.3056	0.3034	0.3011	0.2989	0.2966	0.2943	0.2920
0.8	0.2897	0.2874	0.2850	0.2827	0.2803	0.2780	0.2756	0.2732	0.2709	0.2685
0.9	0.2661	0.2637	0.2613	0.2589	0.2565	0.2541	0.2516	0.2492	0.2468	0.2444
1.0	0.2420	0.2396	0.2371	0.2347	0.2323	0.2299	0.2275	0.2251	0.2227	0.2203
1.1	0.2179	0.2155	0.2131	0.2107	0.2083	0.2059	0.2036	0.2012	0.1989	0.1965
1.2	0.1942	0.1919	0.1895	0.1872	0.1849	0.1826	0.1804	0.1781	0.1758	0.1736
1.3	0.1714	0.1691	0.1669	0.1647	0.1626	0.1604	0.1582	0.1561	0.1539	0.1518
1.4	0.1497	0.1476	0.1456	0.1435	0.1415	0.1394	0.1374	0.1354	0.1334	0.1315
1.5	0.1295	0.1276	0.1257	0.1238	0.1219	0.1200	0.1182	0.1163	0.1145	0.1127
1.6	0.1109	0.1092	0.1074	0.1057	0.1040	0.1023	0.1006	0.0989	0.0973	0.0957
1.7	0.0940	0.0925	0.0909	0.0893	0.0878	0.0863	0.0848	0.0833	0.0818	0.0804
1.8	0.0790	0.0775	0.0761	0.0748	0.0734	0.0721	0.0707	0.0694	0.0681	0.0669
1.9	0.0656	0.0644	0.0632	0.0620	0.0608	0.0596	0.0584	0.0573	0.0562	0.0551
2.0	0.0540	0.0529	0.0519	0.0508	0.0498	0.0488	0.0478	0.0468	0.0459	0.0449
2.1	0.0440	0.0431	0.0422	0.0413	0.0404	0.0396	0.0387	0.0379	0.0371	0.0363
2.2	0.0355	0.0347	0.0339	0.0332	0.0325	0.0317	0.0310	0.0303	0.0297	0.0290
2.3	0.0283	0.0277	0.0270	0.0264	0.0258	0.0252	0.0246	0.0241	0.0235	0.0229
2.4	0.0224	0.0219	0.0213	0.0208	0.0203	0.0198	0.0194	0.0189	0.0184	0.0180
2.5	0.0175	0.0171	0.0167	0.0163	0.0158	0.0154	0.0151	0.0147	0.0143	0.0139
2.6	0.0136	0.0132	0.0129	0.0126	0.0122	0.0119	0.0116	0.0113	0.0110	0.0107
2.7	0.0104	0.0101	0.0099	0.0096	0.0093	0.0091	0.0088	0.0086	0.0084	0.0081
2.8	0.0079	0.0077	0.0075	0.0073	0.0071	0.0069	0.0067	0.0065	0.0063	0.0061
2.9	0.0060	0.0058	0.0056	0.0055	0.0053	0.0051	0.0050	0.0048	0.0047	0.0046
3.0	0.0044	0.0043	0.0042	0.0040	0.0039	0.0038	0.0037	0.0036	0.0035	0.0034

x	0	1	2	3	4	5	6	7	8	9
3.1	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026	0.0025	0.0025
3.2	0.0024	0.0023	0.0022	0.0022	0.0021	0.0020	0.0020	0.0019	0.0018	0.0018
3.3	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014	0.0013	0.0013
3.4	0.0012	0.0012	0.0012	0.0011	0.0011	0.0010	0.0010	0.0010	0.0009	0.0009
3.5	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007	0.0007	0.0007	0.0006
3.6	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0004
3.7	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003	0.0003	0.0003	0.0003
3.8	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002	0.0002	0.0002	0.0002	0.0002
3.9	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0001	0.0001

Appendix B

The values of the Laplace cumulative distribution function $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}dt$

x	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2356	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3906	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.1049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4274	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4430	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4648	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4700	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4762	0.4757
2.0	0.4772	0.4778	0.4783	0.4788	0.4796	0.4798	0.4803	0.4808	0.4812	0.4817

Appendix B (the end)

x	0	1	2	3	4	5	6	7	8	9
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4874	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4903	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4924	0.4927	0.4929	0.4930	0.1932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4958	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4973
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4980	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4986	0.4986	0.4987	0.4987	0.4988	0.4988	0.4988	0.4989	0.4989	0.4990
3.1	0.4990	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993
3.2	0.4993	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998	0.4998	0.4998	0.4998	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.5000	0.5000	0.5000	0.5000	0.5000
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

Content

НАВЧАЛЬНЕ ВИДАННЯ

ВИЩА ТА ПРИКЛАДНА МАТЕМАТИКА

Методичні рекомендації до практичних завдань з розділу "Прикладна математика" для студентів спеціальності 242 "Туризм" першого (бакалаврського) рівня

(англ. мовою)

Самостійне електронне текстове мережеве видання

Укладач **Місюра** Євгенія Юріївна

Відповідальний за видання *Л. М. Малярець*

Редактор *З. В. Зобова*

Коректор *З. В. Зобова*

Викладено необхідний теоретичний матеріал із навчальної дисципліни та наведено типові приклади, які сприяють найбільш повному засвоєнню матеріалу з розділу "Прикладна математика" та застосуванню здобутих знань на практиці. Подано завдання для самостійної та індивідуальної роботи, перелік теоретичних питань, що сприяють удосконаленню і поглибленню знань студентів із цього розділу.

Рекомендовано для студентів спеціальності 242 "Туризм" першого (бакалаврського) рівня.

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