MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

SIMON KUZNETS KHARKIV NATIONAL UNIVERSITY OF ECONOMICS

ELEMENTARY MATHEMATICS (ALGEBRA)

Textbook

for students of the preparatory department

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S. Kuznets KhNUE
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E43

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All the themes of the school algebra course have been outlined on the basis of the educational program of the secondary school of Ukraine. The methods for solving problems in all topics of algebra have been considered in detail and solutions to typical examples are given. Numerous tasks for the individual solution are offered.

For students of the preparatory department.

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Contents

Introduction

The textbook is intended for foreign students of the preparatory department of Simon Kuznets Kharkiv National University of Economics.

In the process of mastering the course of elementary mathematics, students acquire the skills in solving problems envisaged by the school program in mathematics in Ukraine.

The purpose of the textbook is to remind and expand the basic information on elementary mathematics in terminology and symbolism adopted in Ukraine.

The textbook covers typical examples on all topics, according to the program of the preparatory department. The presentation of the material is accompanied by a detailed explanation of a large number of typical examples; it also demonstrates skills and techniques for solving problems of medium and high complexity, which are indicated by the "*" sign.

Each topic begins with a brief theoretical introduction and a list of the necessary formulas and contains a sufficient number of well-solved typical examples of various degrees of complexity. At the end of each topic the tasks for individual work are given with the answers placed at the end of the book.

The material is selected in such a way that the course of secondary school Mathematics is sufficient to study it.

1. Arithmetic

One of the original concepts of mathematics is the concept of a set. A set itself is an undefined concept.

The objects that make up a set are called its elements. One can record the set A as follows: $A = \{x\}$, where x is the element of the set, which means: $x \in A$.

Suppose we have two sets: $A = \{a\}$ and $B = \{b\}$.

The intersection of A and B, denoted by $A \cap B$, is the set of all elements that are members of both *A* and *B* (Fig. 1.1).

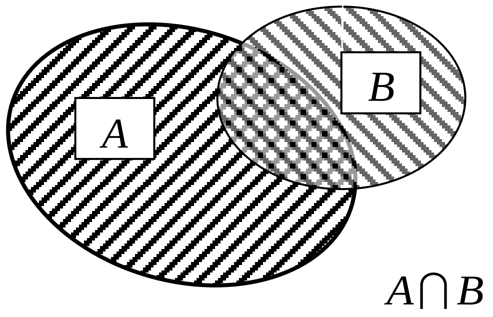


Fig. 1.1. **The Euler – Venn diagram (the intersection of A and B)**

Example 1.1. Let $A = \{1;\ 2;\ 4;\ 8\}$, $B = \{2;\ 5;\ 8\}$. Find $A \cap B$.

Solution: The elements that belong both to set A and set B are these: {2; 8}. Consequently, $A \cap B = \{2;\ 8\}$.

A set that contains no elements is called empty and is denoted by Ø. It is obvious that $A \cap \varnothing = \varnothing$.

For example: $A = \{1;\ 10\}$, $B \in \varnothing$, then $A \cap B = \varnothing$.

The union of two sets, A and B, is the set consisting of all elements entering at least in one of the given sets (Fig. 1.2).

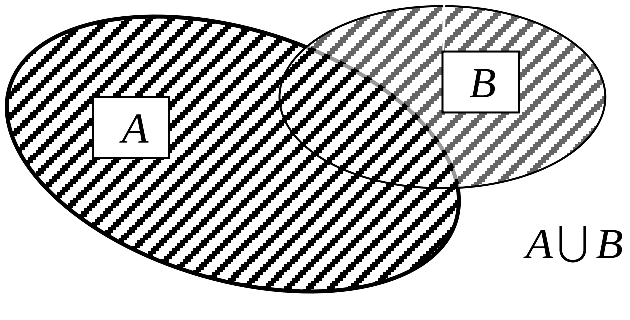


Fig. 1.2. **The Euler – Venn diagram (the union of two sets, A and B)**

Example 1.2. $A=\{1;\ 2;\ 4;\ 7;\ 8\}$, $B=\{2;\ 3;\ 5;\ 8\}$. Then, $A\cup B=\{1;\ 2;\ 3;\ 4;\ 5;\ 7;\ 8\}$.

It is obvious that: $A\cup\varnothing=A$.

The most frequently used numerical sets are sets of natural numbers:

$$N=\{1,\ 2,\ 3,...\}.$$

The set of integer numbers:

$$Z=\{...-3;\ -2;\ -1;\ 0;\ 1,\ 2,\ 3,...\}.$$

The set of rational numbers:

$$Q=\left\{\frac{m}{n},m\in Z,n\in N\right\}.$$

1.2. Natural numbers and operations with them

In the field of natural numbers four operations are defined: addition, subtraction, multiplication, division.

If a and b are natural numbers, then $s=a+b$ is a natural number, a,b are the summands, s is the sum.

For example, $8+3=11$ is a natural number.

$p=a\cdot b$ is a natural number, a,b are the multipliers, p is the product.

For example, $5\cdot 3=15$ is a natural number.

The properties of addition and multiplication of natural numbers

$a+b=b+a$ (the commutative property of addition);

$(a+b)+c=a+(b+c)$ (the associative property of addition);

$a\cdot b=b\cdot a$ (the commutative property of multiplication);

$(a\cdot b)\cdot c=a\cdot(b\cdot c)$ (the associative property of multiplication);

$a\cdot(b+c)=a\cdot b+a\cdot c$ (the distributive property of multiplication over addition).

When subtracting or dividing natural numbers, a natural number is not always obtained.

For example, $5-2=3$ is a natural number;

$4-5=-1$ is not a natural number;

$10:2=5$ is a natural number;

$3:4=\frac{3}{4}$ is not a natural number.

If a, b, c are natural numbers, then $a - b = c$, where: a is the minuend, b is the subtrahend, c is the difference;

$\frac{a}{b} = c$, where: a is the dividend, b is the divisor, c is the quotient.

The order of arithmetic operations in numerical expression is the following. First, the operations in parentheses are performed. In parentheses, one has to multiply and divide first, then add and subtract.

Example 1.3. Calculate:

$$((116 - 16) : 2 + 8 \cdot 20) : 10 - 5 = 16.$$

(1) (2) (3) (4) (5) (6)

Example 1.4. Calculate:

$$(5(25 + 3) - 20 : 4) \cdot 2 - 250 = 20.$$

Divisibility

If the number n can be represented in the form $n = m \cdot k$, then the number n is said to be divisible into whole numbers by m and k, then each of the numbers m and k is called the divisor of the number n.

The number n itself is called the multiple of m and k.

For example, 15 equals 3 times 5; 3 and 5 are divisors of the number 15.

A natural number is said to be whole if it has only two natural divisors: one and this number itself.

A natural number is said to be composite if it has more than two natural divisors. For example, $2, 3, 5, 7, 11, 13, 17$ are whole numbers; $4, 6, 8, 9, 10, 12, 14$ are composite numbers.

The numbers divisible by 2 are called even numbers, while those not divisible by 2 are odd numbers.

Indications of divisibility

The divisibility by 2. A number can be divided by two if the last digit is even or zero. In other cases, it cannot be divided by 2. Or: the natural number is divisible by 2 if and only if its last digit is divisible by 2. For example, 52 734 is divided by 2, because its last digit is 4 – that is, even. 7 693 is not divisible by 2, since 3 is odd. 1 240 is divided by 2, because the last digit is zero.

The divisibility by 3. A number is divisible by 3 if the sum of the digits is divisible by 3. For example, 17 814 can be divided by 3, because the total sum of its digits is 21 and divided by 3.

The divisibility by 4. A natural number is divisible by 4 if and only if its last two digits form a number that is divisible by 4, or two of its last digits are zeros.

Examples:

31 800 can be divided into 4, because it ends in two zeros;

4 846 854 is not divisible by 4 because the last two digits form the number 54, and it is not divisible by 4.

16 604 is divided by 4, because the last two digits 04 form the number 4, which is divided by 4.

The divisibility by 5. A natural number is divided by 5 if and only if the last digit is 0 or 5.

Examples:

245 is a multiple of 5, because the last digit is 5;

774 is not a multiple of 5 because the last digit is not 5 or 0.

The divisibility by 6. A number can be divided by 6, if it can be simultaneously divided into 2 and 3. For example: 216 can be divided into 6, because it is a multiple of two and three.

The divisibility by 7. Remove the last digit, double it, subtract it from the truncated original number and continue doing this until only one digit remains. If this is 0 or 7, then the original number is divisible by 7. For example, to test divisibility of 12 264 by 7, we simply perform the following manipulations: $1226 - 8 = 1218$; $121 - 16 = 105$; $10 - 10 = 0$. Thus, 12 264 is divisible by 7.

The divisibility by 8. If only the last three digits of a whole number are zeros or divisible by 8, then the entire number is divisible by 8.

Examples:

456 000 can be divided by 8, because at the end it has three zeros.

160 004 cannot be divided by 8, because the last three digits form the number 4, which is not a multiple of 8.

111 640 is a multiple of 8, because the last three digits form the number 640, which can be divided by 8.

The divisibility by 9. 9 is a multiple of those numbers whose sum of digits can be divided by 9. For example: The number 111 499 is not divisible by 9, because the sum of the digits (25) is not divided by 9. The number 51 633 can be divided by 9, because the sum of its digits (18) is a multiple of 9.

The divisibility by 10. Only those numbers whose last digit is 0 can be divided by 10.

Examples:
4500 can be divided by 10 and 100.

The decomposition of a natural number into prime factors

Any composite number can be decomposed into prime factors and this can be done in only one way.

Example 1.5. Factorize the numbers $720, 972$.

720	2		972	2
360	2		486	2
180	2		243	3
90	2		81	3
45	3		27	3
15	3		9	3
5	5		3	3
1			1	

$720 = 2^4 \cdot 3^2 \cdot 5$ $\qquad$ $972 = 2^2 \cdot 3^5$

Comment. The product of the same factors is as follows:
$a \cdot a \cdot ... \cdot a = a^n$, where *n* is the power, a is the base, n is the exponent.

The largest common divisor

The largest common divisor (LCD) of several numbers is the largest number by which these numbers are divisible. To find this number it is necessary:

1) to decompose these numbers into the prime factors;
2) to compose a product of common prime factors with the least exponent;
3) to calculate the resulting product.

Example 1.6. Find the LCD of numbers 72 and 180.

1. The decomposition of the numbers into the prime factors:

$$\begin{array}{r|l} 72 & 2 \\ 36 & 2 \\ 18 & 2 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array} \qquad \begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array}.$$

Then, $72 = 2^3 \cdot 3^2$, $180 = 2^2 \cdot 3^2 \cdot 5$.

2. The product of common simple factors with the smallest exponent: $2^2 \cdot 3^2$.

3. The calculated resulting product: $\mathrm{LCD}(72;180) = 36$.

The least common multiple

The least common multiple (LCM) is the smallest natural number that is divided into each of the given numbers.

To find the LCM it is necessary:

1) to decompose the numbers into prime factors;

2) to compose a product of all the resulting simple factors, taking each with a maximal exponent;

3) to find the resulting product.

Example 1.7. Find the LCM of numbers 72 and 180.

1. We decompose the numbers into prime factors:

$$72 = 2^3 \cdot 3^2, \quad 180 = 2^2 \cdot 3^2 \cdot 5.$$

2. We compose the product of all prime factors, taking each with the largest exponent:

$$2^3 \cdot 3^2 \cdot 5.$$

3. We find the value of the product:

$$LCM(72;\ 180) = 360.$$

Comment. The following equality holds:

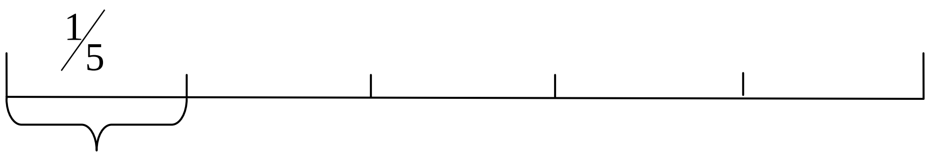
$$ab = LCM(a;b) \cdot LCD(a;b).$$

1.3. Ordinary fractions

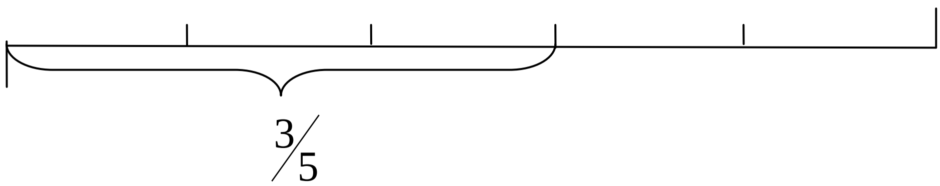
Fractions in which both the numerator and denominator are integers are called simple fractions, common fractions, vulgar fractions or ordinary fractions.

In the fraction $\frac{a}{b}$, a is the numerator, b is the denominator.

For example, $1/5$ means that the unit is divided into 5 equal parts and one part is taken:



$3/5$ means that the unit is divided into 5 equal parts and three parts are taken:



A set of rational numbers is a set of ordinary fractions. If the numerator is less than the denominator, then the fraction is called proper. Also, fractions that are greater than 0 but less than 1 are called proper fractions. In proper fractions, the numerator is less than the denominator. For example: $\frac{2}{3}, \frac{4}{9}$.

When a fraction has a numerator that is greater than or equal to the denominator, the fraction is an improper fraction. An improper fraction is always 1 or greater than 1. For example: $\frac{3}{2}, \frac{5}{3}, \frac{6}{6}$.

And, finally, a mixed number is a combination of a whole number and a proper fraction. For example: $3\frac{2}{3}, 5\frac{3}{8}$.

The main property of a fraction

If the numerator and the denominator of a fraction are multiplied or divided by the same natural number, the fraction does not change:

$$\frac{a}{b}=\frac{a\cdot k}{b\cdot k};\quad \frac{a}{b}=\frac{\frac{a}{k}}{\frac{b}{k}},$$

where k is a natural number. This property allows one to reduce fractions.

For example: $\frac{12}{15}=\frac{4\cdot 3}{5\cdot 3}=\frac{4}{5};\qquad \frac{30}{105}=\frac{2\cdot 3\cdot 5}{3\cdot 5\cdot 7}=\frac{2}{7}.$

Reducing fractions to a common denominator

To bring the fractions to the lowest common denominator, it is necessary:

1) to find the least common multiple of the fractional denominators;

2) to calculate the additional factors to each fraction;

3) to multiply the numerator and denominator of each fraction by the corresponding additional factor.

For example:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd};\quad \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

Addition and subtraction of fractions

When adding (subtracting) fractions with the same denominators, the numerator of the second fraction is added to the numerator of the first fraction (the numerator of the second fraction is subtracted from the numerator of the first fraction) and the same denominator is left.

Example 1.8.

$$\frac{2}{13}+\frac{5}{13}=\frac{2+5}{13}=\frac{7}{13}.$$

Example 1.9.

$$\frac{5}{8}-\frac{2}{8}=\frac{5-2}{8}=\frac{3}{8}.$$

When adding (subtracting) fractions with different denominators, you first need to bring the fractions to a common denominator.

Example 1.10.

$$\frac{7}{12}+\frac{4}{15}=\frac{35+16}{60}=\frac{17}{20}\ .$$

Example 1.11.

$$\frac{7}{12}-\frac{4}{15}=\frac{35-16}{60}=\frac{19}{60}\ .$$

When adding mixed fractions, the integer and fractional parts must be added separately.

Example 1.12.

$$2\frac{1}{4}+3\frac{2}{3}=2+3+\frac{1}{4}+\frac{2}{3}=5+\frac{3+8}{12}=5+\frac{11}{12}=5\frac{11}{12}.$$

When subtracting mixed fractions, two cases are distinguished:

1) The fractional part of the first fraction is greater than or equal to the fractional part of the second fraction; in this case, the whole part of the second fraction is subtracted from the whole part of the first fraction, and the fractional part of the second fraction is subtracted from the fractional part of the first fraction.

Example 1.13.

$$8\frac{7}{12}-5\frac{1}{6}=8-5+\frac{7}{12}-\frac{1}{6}=3+\frac{7-2}{12}=3+\frac{5}{12}=3\frac{5}{12};$$

2) The fractional part of the first fraction is less than the fractional part of the second fraction; in this case, the unit of the whole part of the first fraction is replaced by an equal fraction.

Example 1.14.

$$6\frac{1}{4}-3\frac{2}{3}=6-3+\frac{1}{4}-\frac{2}{3}=3+\frac{3-8}{12}=3-\frac{5}{12}=2\frac{7}{12}.$$

Comment. The addition and subtraction of mixed fractions can be performed by first representing these fractions in the form of improper fractions.

Example 1.15.

$$4\frac{2}{7}+3\frac{1}{5}=\frac{30}{7}+\frac{16}{5}=\frac{150+112}{35}=\frac{262}{35}.$$

Example 1.16.

$$6\frac{1}{4}-2\frac{3}{8}=\frac{25}{4}-\frac{19}{8}=\frac{50-19}{8}=\frac{31}{8}.$$

Multiplication of fractions

When multiplying fractions, the numerator of the first fraction is multiplied by the numerator of the second fraction, and the denominator of the first fraction is multiplied by the denominator of the second fraction. For example:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

Example 1.17.

$$\frac{2}{5}\cdot\frac{3}{8}=\frac{2\cdot 3}{5\cdot 8}=\frac{3}{5\cdot 4}=\frac{3}{20}.$$

Comment. When multiplying mixed fractions, they are previously represented as improper fractions.

Example 1.18.

$$3\frac{3}{4}\cdot 2\frac{2}{9}=\frac{15}{4}\cdot\frac{20}{9}=\frac{3\cdot 5\cdot 4\cdot 5}{4\cdot 3\cdot 3}=\frac{25}{3}.$$

Division of fractions

When dividing fractions, the numerator of the first fraction is multiplied by the denominator of the second fraction, and the denominator of the first fraction is multiplied by the numerator of the second fraction. For example:

$$\frac{a}{b}:\frac{c}{d}=\frac{ad}{bc}.$$

Example 1.19.

$$\frac{5}{8}:\frac{3}{4}=\frac{5\cdot 4}{8\cdot 3}=\frac{5}{6}.$$

Comment. If one or both of the fractions are mixed, then in the case of division, they must first be represented as improper fractions.

Example 1.20.

$$4\frac{3}{5}:2\frac{2}{3}=\frac{23}{5}:\frac{8}{3}=\frac{23\cdot 3}{5\cdot 8}=\frac{69}{40}.$$

Two numbers are called mutually inverse if their product is equal to one. For example:

$$4 \text{ and } \frac{1}{4},\ \frac{2}{3} \text{ and } \frac{3}{2},\frac{c}{d} \text{ and } \frac{d}{c}.$$

Taking into account the mutually inverse numbers, the division of fractions can be reduced to the multiplication of fractions:

$$2\frac{1}{4}:3\frac{2}{3}=\frac{9}{4}:\frac{11}{3}=\frac{9}{4}\cdot\frac{3}{11}=\frac{27}{44}.$$

Example 1.21. Perform the operations:

$$\left(49\frac{5}{24}-46\frac{7}{20}\right):2\frac{1}{3}+\frac{3}{5}.$$

Solution:

1) $49\frac{5}{24}-46\frac{7}{20}=49-46+\frac{5}{24}-\frac{7}{20}=3+\frac{25-42}{120}=3-\frac{17}{120}=$

$=\frac{360-17}{120}=\frac{343}{120};$

2) $\frac{343}{120}:2\frac{1}{3}=\frac{343}{120}:\frac{7}{3}=\frac{343\cdot 3}{120\cdot 7}=\frac{49}{40};$ 3) $\frac{49}{40}+\frac{3}{5}=\frac{49+24}{40}=\frac{73}{40}.$

Example 1.22. Perform the operations:

$$\left(152\frac{3}{4}-148\frac{3}{8}\right)\cdot\frac{3}{10}-\frac{5}{16}.$$

Solution:

1) $152\frac{3}{4}-148\frac{3}{8}=4+\frac{3}{4}-\frac{3}{8}=4+\frac{6-3}{8}=4\frac{3}{8};$

2) $4\frac{3}{8}\cdot\frac{3}{10}=\frac{35}{8}\cdot\frac{3}{10}=\frac{21}{16};$ 3) $\frac{21}{16}-\frac{5}{16}=\frac{16}{16}=1.$

1.4. Decimal fractions

A fraction is called a decimal if its denominator (the bottom number) is a power of ten (such as 10, 100, 1000, etc.). For example:

$$\frac{1}{10}=0.1;\quad \frac{5}{100}=0.05;\quad \frac{18}{1000}=0.018.$$

Addition and subtraction of decimals

Addition and subtraction of decimals are performed in the same way as addition and subtraction of natural numbers. To do this, you only need to write down each bit under the same category name.

Multiplication of decimals

To multiply one decimal fraction by another, it is necessary to multiply them, not paying attention to the decimal points and then to separate with the decimal point as many decimal places as there are in both factors.

Division of decimals

The division of a decimal by a natural number is carried out in the same way as the division of a natural number by a natural number, with the decimal point put after the division of the whole part is completed.

For instance, for 938.4:8 do the following:

```
_938.4 | 8
  8    | 117.3
  --
 _13
   8
   --
  _58
   56
   --
   _24
    24
    --
     0
```

The division can be performed directly or by increasing the dividend and divisor by 10, 100, 1000, … times. For instance,

$$2.28:3, \text{ or } 228:300=0.76.$$

When you divide a decimal by a decimal, you must move the decimal point to the right in the divisible and divisors by as many digits as there are after the decimal point in the divisor, and perform the division. For instance,

$$13.886:0.212=13886:212=65.5.$$

In division, the final decimal fraction is not always obtained. For instance,

$$2:0.3=0.66666\ldots.$$

In this case, they are converted into ordinary fractions, that is $\frac{2}{0.3}=\frac{20}{3}$.

1.5. Conversion of decimal fractions into ordinary ones and vice versa

To convert a decimal into an ordinary fraction, you need to write in the numerator the number of the fraction after the decimal point, and in the denominator – one with zeros to be as many as the digits in the number to the right of the decimal point.

For instance,

$$0.7=\frac{7}{10};\quad 0.21=\frac{21}{100};\quad 0.031=\frac{31}{1000}.$$

If the decimal fraction is greater than one, then only the fractional part should be converted into the ordinary fraction.

For instance,

$$3.8=3\frac{8}{10}=3\frac{4}{5}.$$

To convert an ordinary fraction into a decimal one, you need to divide the numerator by the denominator. As a result, we get a finite or infinite decimal fraction. For instance,

$$\frac{1}{4}=0.25,\quad \frac{1}{3}=0.333...,\quad \frac{2}{15}=0.1333....$$

An infinite decimal fraction, in which, starting with a certain digit, the numbers are repeated, is called periodic.

A repeating group of digits after the decimal point is called the period. For instance, for the fraction $0.333...$ the period is 3; for $2.1878787...$ the period is 87.

The short entry of these numbers is: $0.(3);\ 2.1(87)$.

If the period begins immediately after the decimal point, then such a fraction is called purely periodic. For instance, $0.6767...=0.(67)$.

If there are decimals between the decimal point and the period, the fraction is called a mixed periodic fraction.

For instance, $2.16666... = 2.1(66)$.

1.6. The inversion of a periodic fraction into an ordinary one

A purely periodic fraction is equal to such an ordinary fraction, in which the numerator is a period, and the denominator contains as many nines as the digits in the period.

For instance,

$$0.(7) = \frac{7}{9}; \quad 0.(13) = \frac{13}{99}.$$

A mixed periodic fraction is equal to such an ordinary fraction, in the numerator of which there is a difference between the number standing up to the second period and the number standing up to the first period, and in the denominator there are as many 9s as the digits in the period and as many zeros after them as the digits between the decimal point and the period.

For instance,

$$0.4(61) = \frac{461-4}{990} = \frac{457}{990}; \qquad 3.41(3) = 3\frac{413-41}{900} = 3\frac{372}{900} = 3\frac{31}{75}.$$

Do the following tasks:

Example 1.23.

$$\frac{0.85 - 1.2 \cdot \frac{2}{3}}{2\frac{1}{3} \cdot 9 - 20} = \frac{0.85 - 0.8}{\frac{7}{3} \cdot 9 - 20} = \frac{0.05}{1} = 0.05.$$

Example 1.24.

$$\frac{\left(1\frac{1}{12} + 2\frac{5}{32} + \frac{1}{24}\right) \cdot 9\frac{3}{5} + 2.13}{0.4}.$$

Solution:

1) $1\frac{1}{12}+2\frac{5}{32}+\frac{1}{24}=3+\frac{16+30+8}{192}=3\frac{54}{192}=3\frac{9}{32}$;

2) $3\frac{9}{32}\cdot 9\frac{3}{5}=\frac{105}{32}\cdot\frac{48}{5}=\frac{105\cdot 48}{32\cdot 5}=\frac{21\cdot 3}{2}=\frac{63}{2}$;

3) $\frac{63}{2}+2.13=31.5+2.13=33.63$;

4) $33.63:0.4=84.075$.

Example 1.25.

$$\left(9\frac{3}{20}-1{,}24\right):2\frac{1}{3}+\left(\frac{3}{4}+2\frac{5}{8}\right):0.625.$$

Solution:

1) $1.24=1\frac{24}{100}=1\frac{6}{25}$;

2) $9\frac{3}{20}-1\frac{6}{25}=8+\frac{3}{20}-\frac{6}{25}=8+\frac{15-24}{100}=8-\frac{9}{100}=\frac{800-9}{100}=\frac{791}{100}$;

3) $\frac{791}{100}:2\frac{1}{3}=\frac{791\cdot 3}{100\cdot 7}=\frac{113\cdot 3}{100}=\frac{339}{100}$;

4) $\frac{3}{4}+2\frac{5}{8}=2+\frac{6+5}{8}=2+\frac{11}{8}=3\frac{3}{8}$;

5) $3\frac{3}{8}:\frac{625}{1000}=\frac{27}{8}\cdot\frac{1000}{625}=27\cdot\frac{125}{625}=\frac{27}{5}$;

6) $\frac{339}{100}+\frac{27}{5}=\frac{339+540}{100}=\frac{879}{100}=8.79$.

Example 1.26.

$$\frac{\left(0.(74)+0.2(3)+\frac{26}{495}\right):\frac{1}{60}}{\frac{1}{8}\cdot 2.20(4)}.$$

Solution:

1) $0.(74)=\frac{74}{99}$; 2) $0.2(3)=\frac{23-2}{90}=\frac{21}{90}=\frac{7}{30}$;

3) $2.20(4)=2\frac{204-20}{900}=2\frac{184}{900}=2\frac{46}{225}$;

4) $\frac{74}{99}+\frac{7}{30}+\frac{26}{495}=\frac{3700+1155+260}{4950}=\frac{5115}{4950}=\frac{31}{30}$;

5) $\frac{31}{30}:\frac{1}{60}=\frac{31\cdot 60}{30}=62$;

$\frac{1}{8}\cdot 2\frac{46}{225}=\frac{1\cdot 496}{8\cdot 225}=\frac{62}{225}$;

$62:\frac{62}{225}=\frac{62\cdot 225}{62}=225$.

1.7. Real numbers

An irrational number is a number that cannot be represented as a ratio of two integers. The irrational number can be written in the form of an infinite non-periodic decimal fraction. For instance,

$$\sqrt{2},\sqrt{3},\pi,e,...$$

All rational and irrational numbers form a set of real numbers. It is denoted by R. The set R can be represented on a numerical axis (Fig. 1.3).

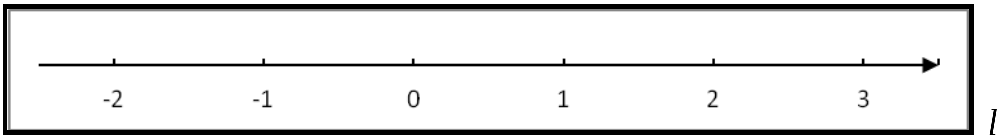


Fig. 1.3. **A numerical axis**

The absolute value of a number is

$$|a| = \left[\begin{array}{l} a, \;\; a \geq 0, \\ -a, \;\; a < 0. \end{array} \right.$$

For instance, $|5| = 5$, because of $5 > 0$.

$$|-3| = -(-3) = 3, \text{ because of } -3 < 0.$$

$$\left|\sqrt{5} - 1\right| = \sqrt{5} - 1; \quad \left|\sqrt{5} - 5\right| = 5 - \sqrt{5}.$$

Module properties

The main properties are as follows:

$$|a| \geq 0; \qquad |a| \geq a; \qquad |a| = |-a|; \qquad |a \cdot b| = |a| \cdot |b|; \qquad |a|^2 = a^2;$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}; \qquad |a + b| \leq |a| + |b|; \quad |a - b| \geq |a| - |b|.$$

Geometrically, the modulus of the number a is the distance from the origin to the point to which the given number corresponds.

For instance, two points on the number line $a_1 = 2$ and $a_2 = -2$ correspond to the expression $|a| = 2$ (Fig. 1.4).

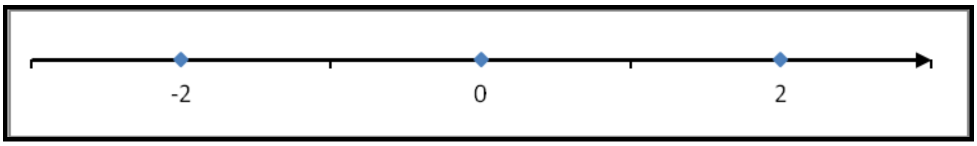


Fig. 1.4. **The modulus of the number *a* on the numerical axis**

Example 1.27. The expression $|a| < 3$ represented on a numerical axis is shown in Fig. 1.5.

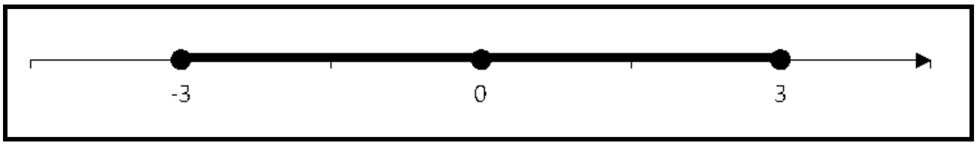


Fig. 1.5. $|a| < 3$ **on the numerical axis**

Example 1.28. The expression $|a| \geq 1$ represented on a numerical axis is shown in Fig. 1.6.

We have:

$$\left[\begin{array}{l} a \geq 1 \\ a \leq -1 \end{array}\right. \qquad a \in (-\infty; -1] \cup [1; \infty).$$

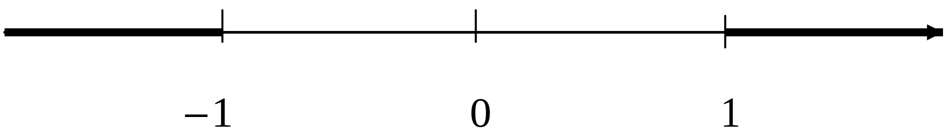


Fig. 1.6. $|a| \geq 1$ **on the numerical axes**

1.8. Relations and proportions

The quotient of two numbers a and b is called the ratio of these numbers:

$$\frac{a}{b} \equiv a/b.$$

The equality of two relations is called the proportion, that is

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow \quad a : b = c : d,$$

where a and d are the extreme terms of the proportion, b and c are the inner terms of the proportion.

The main property of proportions

The product of the inner terms is equal to the product of the extreme terms:

$$\frac{a}{b}=\frac{c}{d}\Rightarrow a\cdot d=b\cdot c.$$

Example 1.29. Find x in the proportion:

$$\frac{x}{12}=\frac{3}{4}.$$

Solution:

$$\frac{x}{12}=\frac{3}{4} \quad \Leftrightarrow \quad 4x=3\cdot 12\Rightarrow \quad x=\frac{3\cdot 12}{4}=9.$$

Example 1.30. Construct proportions from equations: 1) $12\cdot 6=8\cdot 9$; 2) $5y=3x$.

Solution:

1) $\frac{12}{8}=\frac{9}{6};\ \frac{12}{9}=\frac{8}{6};\ \frac{6}{8}=\frac{9}{12};\ \frac{6}{9}=\frac{8}{12};\ \frac{8}{12}=\frac{6}{9};\ \frac{9}{6}=\frac{12}{8}.$

2) $\frac{5}{3}=\frac{x}{y};\ \frac{5}{x}=\frac{3}{y};\ \frac{y}{x}=\frac{3}{5};\ \frac{x}{y}=\frac{5}{3};\ \frac{3}{5}=\frac{y}{x};\ \frac{3}{y}=\frac{5}{x}.$

Direct and inverse proportional relationships between the quantities

Two quantities are called directly proportional if the increase (decrease) in one of them by several times entails the increase (decrease) in the other at the same time:

$$y=kx\Rightarrow\frac{y_1}{y_2}=\frac{x_1}{x_2}.$$

Two quantities are called inversely proportional if the increase (decrease) in one of them by several times entails a decrease (increase) in the other at the same time:

$$y = \frac{k}{x} \Rightarrow \frac{y_1}{y_2} = \frac{x_2}{x_1}.$$

Example 1.31. Out of 21 kg of raw coffee, 16.8 kg of roasted coffee is obtained. How much raw coffee should you take to get 28 kg of roasted coffee?

Solution:

The raw coffee: The roasted coffee:

The raw coffee:	The roasted coffee:	
21	16.8	The direct proportionality.
x	28	

Forming the proportion:

$$\frac{21}{x} = \frac{16.8}{28},$$

from which we have:

$$x = \frac{28 \cdot 21}{16.8} = 35 \text{ (kg)}.$$

Answer: 35 kg of raw coffee should be taken.

Example 1.32. 8 workers did some work in 6 days. How many days will 12 workers be doing this work?

Solution:

Number of workers:	Number of days:	
8	6	The inverse proportionality.
12	x	

Forming the proportion: $\frac{8}{12} = \frac{x}{6}$, from which: $x = \frac{8 \cdot 6}{12} = 4$ (days).

Example 1.33. In the goldsmith's workshop, gold and silver were alloyed in the ratio of 2.4 : 6. Silver made 33 grams. How much gold was taken?

Solution: Forming the proportion:

$$\frac{x}{33}=\frac{2.4}{6},$$

wherefrom:

$$x=\frac{33\cdot 2.4}{6}=13.2\text{(g)}.$$

Answer: 13.2 g of gold was taken.

Example 1.34. To get 4.2 kg of butter, you need to have 9.45 buckets of milk. How many buckets of milk should one take to get 10 kg of butter?

Solution: We have a directly proportional relationship. Forming the proportion:

$$\frac{4.2}{9.45}=\frac{10}{x},$$

from where:

$$x=\frac{10\cdot 9.45}{4.2}=22.5\text{(buckets)}.$$

Answer: 22.5 buckets of milk are required to get 10 kg of butter.

Example 1.35. To paper a room, 14 pieces of wallpaper 56 cm width were used. How many pieces of wallpaper of the same length are needed to paper this room, if the width of the wallpaper is 49 cm?

Solution: With a smaller width of the wallpapers, more pieces will be required, that is, we have an inverse proportional relationship.

Forming the proportion:

$$\frac{14}{x}=\frac{49}{56}.$$

From this we have:

$$x=\frac{14\cdot 56}{49}=16.$$

Answer: 16 pieces of wallpaper are required to paper this room.

Proportional division

To divide the number x in proportion to the given numbers ($a:b:c$), you need to divide this number by the sum of the given numbers $(a+b+c)$ and multiply the result by each of them ($a, b\ or\ c$).

Let

$$x = x_1 + x_2 + x_3,$$

and

$$x_1 : x_2 : x_3 = a : b : c \equiv \frac{x_1}{a} = \frac{x_2}{b} = \frac{x_3}{c}.$$

Then:

$$\frac{x_1}{a} = \frac{x_2}{b} = \frac{x_3}{c} = k \Rightarrow x_1 = ka; \quad x_2 = kb; \quad x_3 = kc,$$

where, k is the coefficient of proportionality. Then:

$$x = ka + kb + kc = k(a+b+c), \quad k = \frac{x}{a+b+c},$$

After this we can find the needed x_1, x_2, x_3 as: $x_1 = ka$; $x_2 = kb$; $x_3 = kc$.

Example 1.36. The number 510 is to be divided into parts directly proportional to the numbers 3, 5 and 7.

Solution: Find the coefficient of proportionality:

$$\frac{x_1}{3} = \frac{x_2}{5} = \frac{x_3}{7} = k;$$

under condition

$$510 = x_1 + x_2 + x_3.$$

From this:

$$510 = 3k + 5k + 12k, \Rightarrow 510 = 15k, \Rightarrow k = 510/15 = 34.$$

Then,

$$x_1 = 3 \cdot 34 = 102, \quad x_2 = 5 \cdot 34 = 170, \quad x_3 = 7 \cdot 34 = 238.$$

To divide a number into parts inversely proportional to the given numbers, it is sufficient to divide this number into parts directly proportional to the inverse of the given numbers.

Example 1.37. The volume of three rooms is 6860 m^3. What is the volume of each room, if their volumes are inversely proportional to the numbers 2/3; 0.75; 0.8?

Solution: Let us find the reciprocal numbers: $\frac{3}{2}$ is inverse for $\frac{2}{3}$; $\frac{4}{3}$ is inverse for $0.75 = \frac{3}{4}$; $\frac{5}{4}$ is inverse for $0.8 = \frac{4}{5}$. We form further the equality:

$$6860 = \frac{3}{2}k + \frac{4}{3}k + \frac{5}{4}k \Rightarrow 6860 = \frac{18+16+15}{12}k \Rightarrow 6860 = \frac{49}{12}k \Rightarrow k = 1680.$$

Then the volume of the first room is $\frac{1680 \cdot 3}{2} = 2520$ m^3;

the volume of the second room is $\frac{1680 \cdot 4}{3} = 2240$ m^3;

the volume of the third room is $\frac{1680 \cdot 5}{4} = 2100$ m^3.

1.9. Percent. The main problems with percent

Percent (%) of the number is the hundredth part of this number.

Three main types of tasks are considered:

1. Finding percentages of numbers:

$$\begin{matrix} 100\,\% \to A \\ p\,\% \to x \end{matrix} \quad \Rightarrow \quad x = \frac{A \cdot p}{100}.$$

2. Finding the number by the given value of its percentage:

$$\begin{array}{l} A \to p\,\% \\ x \to 100\,\% \end{array} \Rightarrow x = \frac{A \cdot 100}{p}.$$

3. Finding the percentage of two numbers:

$$\begin{array}{l} A \to 100\,\% \\ B \to x\,\% \end{array} \quad \Rightarrow \quad x = \frac{B \cdot 100\,\%}{A}$$

Example 1.38. Find $12\,\%$ of 15.

Solution:

$$\begin{array}{l} 100\,\% \to 15 \\ 12\,\% \to x \end{array} \Rightarrow x = \frac{15 \cdot 12}{100} = 1.8.$$

Example 1.39. Find the number 10 % of which is 80.
Solution:

$$\begin{array}{l} 10\,\% \to 80 \\ 100\,\% \to x \end{array} \Rightarrow x = \frac{80 \cdot 100}{10} = 800.$$

Example 1.40. What percentage is 300 of 400?
Solution:

$$\begin{array}{l} 400 \to 100\,\% \\ 300 \to x \end{array} \Rightarrow x = \frac{300 \cdot 100\,\%}{400} = 75\ \%.$$

Example 1.41. Having worked a third of the shift, the worker increased labor productivity by 10 %. After working 2/3 of the shift, the worker increased labor productivity by another 10 % of the initial productivity. During the entire shift, the worker produced 264 parts. How many parts did he make before raising productivity?

Solution: Let the productivity of labor before its increase be denoted by P. Then in the second period of the shift the productivity of labor is 1.1 P, and in the third period of the shift, it is 1.2 P. Let x_1 be the number of parts manufactured in the first period of the shift; x_2 – the number of parts manufactured in the second period of the shift; x_3 – the number of parts manufactured in the third period of the shift. The time of the whole shift is denoted as a unity. Then, $x_1 = P/3$; $x_2 = 1.1P/3$; $x_3 = 1.2P/3$. We know that: $x_1 + x_2 + x_3 = 264$. So, $(1+1.1+1.2)P/3 = 264 \Rightarrow P = 264/1.1 = 240$.

Knowing the productivity of labor in the first period of work, we find the number of parts manufactured in the first period: $x_1 = P/3 = 240/3 = 80$.

Answer: Before the increase in labor productivity, the worker produced 80 parts.

Example 1.42. The first plant produced 350 machines a year more than the second one. 20 % of machines produced by the first plant make 170 units fewer than 40 % of the machines produced by the second plant. How many machines did each plant produce?

Solution: Let the number of machines produced by the second plant be denoted by *x*, then the first plant produced $x + 350$ machines. 20 % of $x + 350$ is $0{,}2(x + 350)$; 40 % of *x* is $0{,}4x$.

By condition: $0{,}2(x+350)+170 = 0{,}4x$, then $0{,}2x + 70 + 170 = 0{,}4x$. From the last equation we have: $x = 1200$.

Answer: the second plant produced $x = 1200$ machines, the first plant produced $x + 350$ machines, that is 1550 machines.

Example 1.43. The farmer sowed 500 hectares of land in 4 days. On the first day, 18 % of the total area was sown, on the second day – $14/41$ of the remainder, and the areas sown on the third and fourth days are proportional to the numbers $2\,{}^1\!/_6$ and $2\,{}^1\!/_3$. How many hectares of land were sown in each of the four days?

Solution: On the first day, 18 % of the total area was sown, that is, 0.18 · 500 = 90 ha. On the second day $14/41$ of the remainder, namely, $(500-90)\cdot 14/41 = 140$ ha, was sown. The remaining area to be sown is 500 − 90 − 140 = 270 ha.

The areas sown on the third and fourth days (x_3 and x_4 correspondently) are proportional to the numbers $2\,{}^1\!/_6$ and $2\,{}^1\!/_3$. That is:

$$\frac{x_3}{2\frac{1}{6}} = \frac{x_4}{2\frac{1}{3}} \quad \Rightarrow \quad \frac{x_3}{\frac{13}{6}} = \frac{x_4}{\frac{7}{3}} = k; \quad \Rightarrow \quad x_3 = \frac{13}{6}k; \quad x_4 = \frac{7}{3}k.$$

Taking into account that $270 = x_3 + x_4$, we have:

$$270 = \frac{13}{6}k + \frac{7}{3}k,\ 270 = \frac{13+14}{6}k \quad \Rightarrow 270 = \frac{27}{6}k \Rightarrow k = 60.$$

Then,

$$x_3 = \frac{13}{6}\cdot 60,\ x_3 = 130; \qquad x_4 = \frac{7}{3}\cdot 60,\ x_4 = 140.$$

Answer: On the first day 90 hectares were sown, on the second day 140 hectares, on the third day 130 hectares, on the fourth day 140 hectares were sown.

Example 1.44. Having overfulfilled the task by 6 %, the turner has machined 159 parts. How many parts should the turner machine in a shift?

Solution:

$$\begin{array}{l} 106\,\% - 159 \\ 100\,\% - x \end{array} \quad \Rightarrow x = \frac{159\cdot 100}{106} = 150.$$

Answer: 150 parts.

Example 1.45. The workshop employs 60 workers, 42 of them with secondary specialized education. What is the percentage of workers with secondary specialized education out of the total number of workers?

Solution:

$$\begin{array}{l} 60 - 100\,\% \\ 42 - x\,\% \end{array} \quad \Rightarrow \quad x = \frac{42\cdot 100\,\%}{60} = 70\,\%.$$

Answer: $70\,\%$.

Accrual of the deposit amount by a simple interest formula

Let the initial deposit in a bank equal a (conventional units). $p\,\%$ a year is accrued on the initial deposit. Calculate the deposit amount in n years.

Solution: In n years the amount of the deposit will be $S = a\cdot\left(1 + n\frac{p}{100}\right)$.

Example 1.46. A client made a bank deposit of 10 000 UAH. Calculate the deposit amount in 10 years if the interest is 4 % per year.

Solution: Using the formula $S = a \cdot \left(1 + n\frac{p}{100}\right)$ we have:

$$S = 10\,000 \cdot \left(1 + 10 \cdot \frac{4}{100}\right) = 14\,000 \text{ UAH.}$$

Answer: $14\,000$ UAH.

Calculation of the deposit amount by the compound interest formula

Let the initial deposit in a bank equal a (conventional units). During the year, $p\,\%$ of the accumulated amount is accrued. Calculate the deposit amount in n years.

In n years the amount of the deposit will be:

$$S = a \cdot \left(1 + \frac{p}{100}\right)^n .$$

Example 1.47. A client made a bank deposit of 10 000 UAH. The yearly interest is 7 %. Calculate the deposit amount in 10 years.

Solution: We use the same formula: $S = a \cdot \left(1 + \frac{p}{100}\right)^n$.

Then: $S = 10\,000 \cdot \left(1 + \frac{7}{100}\right)^{10} = 19\,671{,}5$ UAH.

Answer: $19\,671{,}5$ UAH.

Tasks for individual work

1.48. Decompose the numbers into prime factors:

1) 216; 162; 144; 225; 512; 675; 1024.
2) 60; 180; 220; 350; 400; 1200.
3) 875; 2025; 2376; 3969.

1.49. Find the greatest common divisor of the numbers:
1) 12 and 18; 2) 50 and 175; 3) 675 and 825; 4) 7200 and 1080;
5) 324, 144 and 432;   6) 320, 640 and 840.
1.50. Find the least common multiple of the numbers:
1) 72 and 99; 2) 425 and 510; 3) 32, 36, 48 and 72; 4) 212, 318 and 530.

Do the operations:

1.51. $\dfrac{28\frac{4}{5}:13\frac{5}{7}+6\frac{3}{5}:\frac{2}{3}}{1\frac{11}{16}:2\frac{1}{4}}$.

1.52. $\dfrac{4-\left(\frac{9}{10}-\frac{4}{15}\right):3\frac{1}{6}}{8-1\frac{22}{23}\cdot\left(\frac{14}{15}-\frac{1}{6}\right)}+\dfrac{27}{65}$.

1.53. $\left(\frac{1}{2}+0{,}8-1\frac{1}{2}:2{,}5\right):\left(3+4\frac{3}{25}-0{,}12\right)$.

1.54. $\dfrac{\left(6\frac{3}{5}-3\frac{3}{14}\right)\cdot 5\frac{5}{6}}{(21-1{,}25):2{,}5}$.

1.55. $\dfrac{\left(6{,}8-3\frac{3}{5}\right)\cdot 5\frac{5}{6}}{\left(3\frac{2}{3}-3\frac{1}{6}\right)\cdot 56}-27\frac{1}{6}$.

1.56. $\dfrac{\left(13{,}75+9\frac{1}{6}\right)\cdot 1{,}2}{\left(10{,}3-8\frac{1}{2}\right)\cdot\frac{5}{9}}$.

1.57. $\dfrac{\left(12\frac{2}{3}-10\frac{7}{9}\right)\cdot 1\frac{10}{17}}{\left(4-2\frac{1}{3}\right):4\frac{4}{9}}+\dfrac{\left(\frac{1}{5}+0{,}75\right):\frac{1}{20}}{1-\frac{80}{99}}$.

1.58. $\dfrac{\left(0{,}(6)+\frac{1}{3}\right):0{,}125}{0{,}12(3):0{,}0925}$.

1.59. $\dfrac{\left(15\frac{4}{45}-14\frac{4}{15}\right):\frac{1}{30}}{8:6}-\dfrac{4{,}25:0{,}85+1:0{,}5}{(10{,}56-9{,}06)\cdot\frac{1}{3}}$.

1.60. Find x in the proportions:

1) $20:x=25:5$.   2) $\frac{2}{3}:x=\frac{11}{5}:3$.   3) $x:\frac{9}{2}=4:\frac{3}{2}$;

4) $55:8=11:4x$;   5) $\frac{21}{2}:4=\frac{1}{5}x:\frac{1}{7}$;   6) $20:3x=\frac{4}{3}:\frac{1}{10}$.

1.61. The number 3600 is divided proportionally into the numbers $5,7,9,4$. Find the smallest of the numbers obtained.

1.62. The number 798 is divided proportionally into the numbers $2/3;\ 0{,}75;0{,}8$. Find the largest of the numbers obtained.

1.63. The number 228 is divided into three parts inversely proportional to the numbers $1.2\,;4\,;6$. Find the largest of the numbers obtained.

1.64. The number 80 is divided into three parts inversely proportional to the numbers $3;4;12$. Find the largest of the numbers obtained.

1.65. 120 people can do some work in 35 days. How many people more are needed to perform this work in 21 days?

1.66. In a farm, there are 420 cows, for which hay is stored for 240 days. How many days will this hay last, if the herd increases by 60 cows?

1.67. Having exceeded the task by 6 %, the turner has machined 159 parts. How many parts should the turner machine in a shift?

1.68. During 1 hour, a machine tool produced 240 parts. After the reconstruction of the machine, it began to manufacture 288 such parts per hour. What is the percentage of the labor productivity increase?

1.69. At a factory, 35 % of all workers are women, and the rest are men, which is 252 more than women. Determine the total number of workers.

1.70. When selling goods for 1386 UAH, a 10 % profit is received. Determine the prime cost of goods.

1.71. Raisins obtained by drying the grapes make 32 % of the whole weight of the grapes. How much grapes (kg) will give 2 kg of raisins?

1.72. A 72 % piece of the rail length was cut off. The weight of the remaining piece is 45.2 kg. How many kg, with accuracy within a tenth, does the cut piece weigh?

1.73. Determine the percentage of salt in the solution if 300 g of the solution contain 285 g of water.

1.74. The sum of two numbers is 90, and one of them is 20 % less than the other. Find the larger number.

1.75. The difference between two numbers is 20, one of them being 25 % smaller than the other. Find the smaller number.

1.76. The plant produced 990 combines instead of 900, scheduled according to the plan. What is the percentage of exceeding the plan?

1.77. A 12 kg piece of the alloy of copper and tin contains 45 % of copper. How much pure tin should be added to this piece for the resulting new alloy to contain 40 % copper?

1.78. Sea water contains 5 % (by weight) of salt. How much fresh water should be added to 40 kg of sea water, for the salt content in the latter to be 2 %?

1.79. Find the number 2 % of which is equal to

$$\frac{\left(\frac{3}{5}\cdot 0{,}12-0{,}66:30\right):0{,}01}{(0{,}576)^2+0{,}576\cdot 0{,}424+9{,}424}.$$

1.80. Find 4.2 % of $\dfrac{2\frac{5}{8}-\frac{2}{3}\cdot 2\frac{5}{14}}{\left(3\frac{1}{12}+4{,}375\right):19\frac{8}{9}}$.

Questions for self-assessment

1. What is the Euler – Venn diagram?
2. What numbers are natural?
3. The properties of addition and multiplication of natural numbers.
4. The divisibility rules.
5. Arithmetic operations with decimals.
6. Real numbers.
7. Proportions.
8. The proportional division.
9. The percent. The interest.

2. Identical transformations of algebraic expressions

2.1. Transformation of rational expressions

Definitions

An algebraic expression is a mathematical expression in which numbers and letters are joined by signs of addition, subtraction, multiplication, division, raising to a rational power, and extracting an arithmetic root. For example:

$$x^2+2xy;\ (a^2-b^2)(a^2+b^2);\ (x+y)^3;\ \frac{x^2-x+1}{x-1};$$

$$x^{\frac{1}{2}}-3y^{\frac{2}{3}};\ \sqrt{a}-\sqrt{b}.$$

A rational expression in which there is no division into an alphabetic expression and extraction of a root is called an integer. For example:

$$x - 3xy^2; \ (a+b)(a-2b).$$

A fractional expression is a rational expression that contains division by an alphabetic expression. For example:

$$\frac{a^3 - 2ab}{b^2 - a^2}.$$

If an algebraic expression uses the extraction of a root of variables (or the raising of a variable to a power with a fractional exponent), then such an expression is called irrational. For example:

$$\sqrt{x} - \sqrt[3]{y}, \qquad \frac{a^{\frac{2}{3}} - b^{\frac{5}{3}}}{a+2b}.$$

The domain (or the range of admissible values) of an algebraic expression is the set of values of variables under which the algebraic expression makes sense.

For example, the expression $5a - \frac{a}{a-5}$ has a sense if $a \neq 5$, that is $a \in R \setminus \{5\}$.

An identity is an equality that is true for all admissible values of the variables included in it.

For example:

$$a^2 - b^2 = (a-b)(a+b); \ (a+b)^2 = a^2 + 2ab + b^2.$$

Two expressions are said to be identically equal if they take the same values for all common admissible values. For example:

$$2x - 4 \text{ and } 2(x-2); \quad \frac{1}{x-2} - \frac{1}{x+2} \text{ and } \frac{4}{x^2 - 4}.$$

The replacement of one such expression by another is called the identity transformation.

The power of a real number with a natural exponent

By definition of a power with a natural exponent, we have:

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n}, n \in N, n > 1.$$

$$a^1 = a, \qquad a \text{ is a number.}$$

The properties of exponents

If $a > 0,$ then

$$a^m \cdot a^n = a^{m+n};$$

$$a^m : a^n = a^{m-n};$$

$$\left(a^m\right)^n = a^{mn}.$$

If $a > 0, b > 0,$ then

$$(ab)^n = a^n b^n; \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

If $a \neq 0,$ then $a^0 = 1, \quad 1^n = 1.$

A power with a negative exponent:

$$a^{-k} = \frac{1}{a^k}.$$

Comment: $(-1)^{2n} = 1, \quad (-1)^{2n+1} = -1.$

Example 2.1. Using the power properties perform the operations:

1) $3x^m \cdot 2x^n = 6x^{m+n}.$
2) $\left(2x^{n+1}\right)^3 = 8x^{3n+3}.$
3) $\left(-2a^3 \cdot b^{m-1}c^2\right)^5 = -32a^{15}b^{5m-5}c^{10}.$

4) $(20a\cdot b^3c)(4a^3b^4)=80a^4b^7c.$

5) $(a^m\cdot b^3):(a^3\cdot b^2)=a^{m-3}b.$

6) $\left(\dfrac{4^{3x}}{4^x}\right)^2=(4^{2x})^2=4^{4x}.$

The addition and product of polynomials

To perform these operations, open the parentheses and make similar members.

Example 2.2. Perform the operations:

1) $8a^2-3ab-2b^2+3+a^2-4ab+6+5a^2+7b^2-9=14a^2-7ab+5b^2;$

2) $8x^3y^2-2xy^2+3y^4-(4x^3y^2-8x^2y+2y^4)=4x^3y^2-2xy^2+8x^2y-y^4;$

3) $(x^2-3x+3)\cdot 5x^2+2x\cdot(x^3-4x+5)=5x^4-15x^3+15x^2+2x^4-8x^2+$
$+10x=7x^4-15x^3+7x^2+10x;$

4) $(y-1)(y^3+y^2+y+1)=y^4+y^3+y^2+y-y^3-y^2-y-1=y^4-1;$

5) $(a^2+7a+3)(a^2-4a+2)=a^4-4a^3+2a^2+7a^3-28a^2+14a+3a^2-$
$-12a+6=a^4+3a^3-23a^2+2a+6.$

Division of a polynomial by another polynomial

To divide a polynomial A by a polynomial B is to find a polynomial P, multiplying which by B yields A (that is, $A=B\cdot P$, where B is a nonzero polynomial). B is the divisor; P is the quotient of the polynomials A and B.

The division of a polynomial by a polynomial with a remainder is determined analogously to the division of natural numbers with a remainder.

To divide a polynomial A by a polynomial B with a remainder means to find two polynomials P and R such that $A=B\cdot P+R$, with the degree of R less than the degree of the divisor B.

It is convenient to perform the division of polynomials as well as division of numbers by the "corner" method.

Example 2.3. Perform division:

1) Divide x^3-3x^2+4x-4 by $x-2$.

Solution:

$$\begin{array}{l|l}
x^3-3x^2+4x-4 & \dfrac{x-2}{x^2-x+2} \\
\underline{x^3-2x^2} & \\
\quad -x^2+4x & \\
\quad \underline{-x^2+2x} & \\
\qquad\quad 2x-4 & \\
\qquad\quad \underline{2x-4} & \\
\qquad\qquad 0 &
\end{array}.$$

Then, $\dfrac{x^3-3x^2+4x-4}{x-2}=x^2-x+2$.

2) Divide $x^4+2x^3-13x^2-14x+24$ by $x+4$.

Solution:

$$\begin{array}{l|l}
x^4+2x^3-13x^2-14x+24 & \dfrac{x+4}{x^3-2x^2-5x+6} \\
\underline{x^4+4x^3} & \\
\quad -2x^3-13x^2 & \\
\quad \underline{-2x^3-8x^2} & \\
\qquad -5x^2-14x & \\
\qquad \underline{-5x^2-20x} & \\
\qquad\qquad 6x+24 & \\
\qquad\qquad \underline{6x+24} & \\
\qquad\qquad\quad 0 &
\end{array}$$

Then, $\dfrac{x^4+2x^3-13x^2-14x+24}{x+4}=x^3-2x^2-5x+6$.

3) Divide $x^5+5x^2-12x+6$ by x^3+x+5.

Solution:

$$\begin{array}{l|l}
x^5+5x^2-12x+6 & \dfrac{x^3+x+5}{x^2-1} \\
\underline{x^5+x^3+5x^2} & \\
\quad -x^3-12x+6 & \\
\quad \underline{-x^3-x-5} & \\
\qquad -11x+11 &
\end{array}$$

Then, $\dfrac{x^5+5x^2-12x+6}{x^3+x+5}=x^2-1+\dfrac{-11x+11}{x^3+x+5}$.

The formulas of abridged multiplication

$$(a+b)^2=a^2+2ab+b^2;$$
$$(a-b)^2=a^2-2ab+b^2;$$
$$a^2-b^2=(a-b)(a+b);$$
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3;$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3;$$
$$a^3+b^3=(a+b)(a^2-ab+b^2);$$
$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

Example 2.4. Do the operations using the formulas of abridged multiplication:

1) $(3a+2b)^2=9a^2+12ab+4b^2$;

2) $(x-2y)^2=x^2-4xy+4y^2$;

3) $(2a^n-3b^m)^2=4a^{2n}-12a^nb^m+9b^{2m}$;

4) $(x-2)(x^2+2x+4)=x^3-8$;

5) $(1-3a)(1+3a)(1-9a^2)=(1-9a^2)^2=1-18a^2+81a^4$;

6) $(25x^4-9):(5x^2+3)=\dfrac{(5x^2-3)\cdot(5x^2+3)}{5x^2+3}=5x^2-3$;

7) $\left(\dfrac{1}{27}x^3-\dfrac{1}{8}y^3\right):\left(\dfrac{1}{3}x-\dfrac{1}{2}y\right)=\dfrac{\left(\frac{1}{3}x-\frac{1}{2}y\right)\left(\frac{1}{9}x^2+\frac{1}{6}xy+\frac{1}{4}y^2\right)}{\frac{1}{3}x-\frac{1}{2}y}=$

$=\dfrac{1}{9}x^2+\dfrac{1}{6}xy+\dfrac{1}{4}y^2$;

8) $(2^{2n}-5^{2n}):(2^n+5^n)=\dfrac{(2^n-5^n)\cdot(2^n+5^n)}{2^n+5^n}=2^n-5^n$;

9) $\left(3a-\frac{2}{a}\right)^3=27a^3-54a+36\frac{1}{a}-\frac{8}{a^3}$.

Example 2.5. For which values of the parameters A, B and C is the polynomial $P(x)$ identically equal to the polynomial $F(x)$, where

$$P(x)=Ax^2(x+1)+B(x^2+1)(x-6)+Cx(x^2+1),$$
$$F(x)=x^2+5x+6?$$

Solution: The polynomials $P(x)$ and $F(x)$ are identically equal if they have equal coefficients for x^2, x^1, x^0. Therefore we have the system:

$$\begin{array}{ll} x^2: A-6B=1, & A=-5, \\ x^1: B+C=5, & C=6, \\ x^0: -6B=6, & B=-1. \end{array}$$

Example 2.6. Convert the expressions so that they do not contain negative indicators:

1) $2a^{-3}b=\frac{2b}{a^3}$; 2) $5a^{-2}b^{-1}=\frac{5}{a^2b}$;

3) $\frac{(a-b)^{-1}a^2}{b(a+b)}=\frac{a^2}{b(a^2-b^2)}$; 4) $\frac{3x^2}{x^{-1}y^{-2}}=3x^3y^2$.

Example 2.7. Bring out the common multiplier beyond the brackets.

1) $a^{10}+3a^8=a^8(a^2+3)$; 2) $a^{2n-1}b^2+a^{n+2}=a^{n+2}(a^{n-3}b^2+1)$;

3) $x^{2n+1}y^4-x^{n-1}y^5=x^{n-1}y^4(x^{n+2}-y)$;

4) $14x^2y^3+42x^2y^2-28xy=14xy(xy^2+3xy-2)$;

5) $3a(4a-1)-ab(4a-1)-b^2(1-4a)=(4a-1)(3a-ab+b^2)$.

Example 2.8. Fulfill factoring:

1) $ax+ay+3x+3y=a(x+y)+3(x+y)=(x+y)(a+3)$;

2) $56x^2-45y-40xy+63x=56x^2-40xy+63x-45y=8x(7x-5y)+$ $+9(7x-5y)=(7x-5y)(8x+9)$;

3) $a^3+a^2b-a^2c-abc=a^2(a+b)-ac(a+b)=(a+b)(a^2-ac)=$
$=a(a+b)(a-c).$

4) $mx^2-nx^2-nx+mx-m+n=x^2(m-n)+x(m-n)-(m-n)=$
$=(m-n)(x^2+x-1).$

Example 2.9. Fulfill factoring:

1) $c^4-2c^3-c^2+2c=c^3(c-2)-c(c-2)=(c-2)(c^3-c)=(c-2)(c-1)\cdot$
$\cdot(c+1)c.$

2) $81x^9+3x^3=3x^3\left(27x^6+1\right)=3x^3\left((3x^2)^3+1\right)=3x^3\left(3x^2+1\right)\cdot\left(9x^4-3x^2+1\right).$

3) $a^3+a^2+4=a^3+8+a^2-4=(a+2)(a^2-2a+4)+(a-2)(a+2)=$
$=(a+2)(a^2-2a+4+a-2)=(a-2)(a^2-a+2).$

4) $(a-x)y^3-(a-y)x^3+(x-y)a^3=ay^3-xy^3-ax^3+x^3y+a^3x-a^3y=$
$=ay(y^2-a^2)-x(y^3-a^3)+x^3(y-a)=ay(y-a)(y+a)-x(y-a)\times$
$\times(y^2+ay+a^2)+x^3(y-a)=(y-a)(ay^2+a^2y-xy^2-axy-a^2x+x^3)=$
$=(y-a)(y^2(a-x)+ay(a-x)-x(a^2-x^2))=(y-a)(a-x)(y^2+ay-ax-$
$-x^2)=(y-a)(a-x)[(y-x)(y+x)+a(y-x)]=(y-a)(a-x)(y-x)[y+x+a].$

2.2. Conversion of fractional expressions

Example 2.10. Find the domain of the function:

$$\frac{x^2+4}{5x-2}.$$

Solution: This expression is a fraction and exists when the denominator is non-zero, that is, when

$$5x-2\neq 0, x\neq\frac{2}{5}.$$

Then the domain is $x\neq 2/5$.

Example 2.11. Find the domain of the function:

$$\frac{2}{(x-9)(x+3)}.$$

Solution: This expression is a fraction and exists when the denominator is non-zero. Then the domain is:

$$x \neq 9,\ x \neq -3 \text{ or } x \in (-\infty;-3) \cup (-3;9) \cup (9;+\infty).$$

Example 2.12. Find the domain of the function:

$$\frac{5+2/x}{x+1}; \quad x \neq 0, \quad x \neq -1.$$

Solution: This expression is a fraction and exists when the denominator is non-zero. Then the domain is: $x \neq 0, \quad x \neq -1$.

Example 2.13. Find the domain of the function:

$$\frac{2}{|a|+2}, \quad a \in R.$$

Solution: The denominator of this fraction does not vanish because of $|a| \geq 0$. Therefore, this fraction exists for any a, that is: $a \in R$.

In the process of transforming algebraic expressions, they move from one algebraic expression to another, a simpler or more convenient one. In this case, use is made of reducing fractions to a common denominator, factoring, formulas for reduced multiplication, and so on.

Example 2.14. Simplify:

$$\frac{6}{3b-a} + \frac{5a+21b}{a^2-9b^2}.$$

Solution:

$$\frac{6}{3b-a}+\frac{5a+21b}{a^2-9b^2}=\frac{6(3b+a)-(5a+21b)}{9b^2-a^2}=$$

$$=\frac{18b+6a-5a-21b}{9b^2-a^2}=\frac{a-3b}{9b^2-a^2}=-\frac{1}{3b+a}.$$

Example 2.15. Simplify:

$$\frac{a^2-b^2}{a-b}-\frac{a^3-b^3}{a^2-b^2}.$$

Solution:

$$\frac{a^2-b^2}{a-b}-\frac{a^3-b^3}{a^2-b^2}=\frac{(a-b)(a+b)}{a-b}-\frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)}=$$

$$=a+b-\frac{a^2+ab+b^2}{a+b}=\frac{(a^2+2ab+b^2)-(a^2+ab+b^2)}{a+b}=\frac{ab}{a+b}.$$

Example 2.16. Simplify:

$$\frac{x}{ax-2a^2}-\frac{2}{x^2+x-2ax-2a}\cdot\left(1+\frac{3x+x^2}{3+x}\right)$$

Solution:

$$\frac{x}{ax-2a^2}-\frac{2}{x^2+x-2ax-2a}\cdot\left(1+\frac{3x+x^2}{3+x}\right)=$$

$$=\frac{x}{ax-2a^2}-\frac{2}{x(x+1)-2a(x+1)}\cdot\frac{3+x+3x+x^2}{3+x}=$$

$$=\frac{x}{ax-2a^2}-\frac{2}{(x+1)(x-2a)}\cdot\frac{x^2+4x+3}{3+x}=$$

$$=\frac{x}{ax-2a^2}-\frac{2(x+1)(x+3)}{(x+1)(x-2a)}\cdot\frac{1}{(3+x)}=\frac{x-2a}{a(x-2a)}=\frac{1}{a}.$$

Example 2.17. Simplify:

$$\left(\frac{a-1}{a^2-2a+1}+\frac{2(a-1)}{a^2-4}-\frac{4(a+1)}{a^2+a-2}+\frac{a}{a^2-3a+2}\right)\cdot\frac{36a^3-144a-36a^2+144}{a^3+27}.$$

Solution:

$$\left(\frac{a-1}{a^2-2a+1}+\frac{2(a-1)}{a^2-4}-\frac{4(a+1)}{a^2+a-2}+\frac{a}{a^2-3a+2}\right)\cdot\frac{36a^3-144a-36a^2+144}{a^3+27}=$$

$$=\left(\frac{a-1}{(a-1)^2}+\frac{2(a-1)}{(a-2)(a+2)}-\frac{4(a+1)}{(a-1)(a+2)}+\frac{a}{(a-1)(a-2)}\right)\times$$

$$\times\frac{36a\ (a^2-4)-36(a^2-4)}{(a+3)(a^2-3a+9)}=$$

$$=\frac{a^2-4+2a^2-4a+2-4a^2+4a+8+a^2+2a}{(a-1)(a^2-4)}\cdot\frac{(a^2-4)\cdot 36(a-1)}{(a+3)(a^2-3a+9)}=$$

$$=\frac{2a+6}{1}\cdot\frac{36}{(a+3)(a^2-3a+9)}=\frac{72}{a^2-3a+9}.$$

2.3. Transformation of irrational expressions

Algebraic expressions are called irrational if the variables included in them are raised to a rational power.

The root of the *n*-th degree of a is a number, the *n*-th power of which is equal to a, that is, $\left(\sqrt[n]{a}\right)^n=a$.

The root of the *n*-th degree is called arithmetic if $a\geq 0$ and $\sqrt[n]{a}\geq 0$.

The identity of the arithmetic root

$$\sqrt{a^2}=|a|,\quad \sqrt[2n]{a^{2n}}=|a|.$$

The basic properties of the roots

If $a \geq 0$ and $b \geq 0$, then:

1) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$;

2) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}, \quad b \neq 0$;

3) $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$;

4) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = a^{\frac{1}{nm}}$;

5) $\sqrt[nk]{a^k} = \sqrt[n]{a}$;

6) $\sqrt[n]{a^n b} = a \cdot \sqrt[n]{b}$ and $\sqrt[2n]{a^{2n} b} = |a| \cdot \sqrt[2n]{b}$, if $a < 0, b \geq 0$;

7) $a \cdot \sqrt[2n]{b} = \sqrt[2n]{a^{2n} b}$, if $a \geq 0, b \geq 0$; $a \cdot \sqrt[2n]{b} = -\sqrt[2n]{a^{2n} b}$, if $a < 0, b \geq 0$;

8) $\sqrt[2n]{ab} = \sqrt[2n]{|a|} \cdot \sqrt[2n]{|b|}$, if $a < 0, b < 0$;

9) $\sqrt[2n]{\dfrac{a}{b}} = \dfrac{\sqrt[2n]{|a|}}{\sqrt[2n]{|b|}}$, if $a < 0, b < 0$; 10) $\sqrt[2n]{a^{2m}} = \sqrt[n]{|a|^m}$, if $a < 0$.

Example 2.18. Take out the multiplier from under the root:

$$a)\ \sqrt[3]{2^5}; \quad b)\ \sqrt{32}; \quad c)\ \sqrt[4]{a^7 \cdot b^{10}}.$$

Solution: Using the properties, we have:

$$a)\ \sqrt[3]{2^5} = \sqrt[3]{2^3 \cdot 2^2} = 2 \cdot \sqrt[3]{2^2};$$

$$b)\ \sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{4^2 \cdot 2} = \sqrt{4^2} \cdot \sqrt{2} = 4 \cdot \sqrt{2};$$

$$c)\ \sqrt[4]{a^7 \cdot b^{10}} = \sqrt[4]{a^4 \cdot a^3 \cdot b^8 \cdot b^2} = a \cdot b^2 \cdot \sqrt[4]{a^3 \cdot b^2}.$$

Example 2.19. Bring in the multiplier under the root:

$$a)\ 2\sqrt[4]{3}; \quad b)\ a^3 \cdot \sqrt[3]{a \cdot b^2}.$$

Solution: Using the properties, we have:

$$a)\ 2\sqrt[4]{3}=\sqrt[4]{2^4}\cdot\sqrt[4]{3}=\sqrt[4]{16\cdot 3}=\sqrt[4]{48};$$

$$b)\ a^3\cdot\sqrt[3]{a\cdot b^2}=\sqrt[3]{a^9}\cdot\sqrt[3]{a\cdot b^2}=\sqrt[3]{a^{10}\cdot b^2}.$$

Example 2.20. Evaluate:

1) $7-\sqrt{3}-\sqrt{(7+\sqrt{3})^2}$; 2) $\sqrt{(5-\sqrt{5})^2}+\sqrt{(5+\sqrt{5})^2}$;

3) $\sqrt{4-2\sqrt{3}}$; 4) $\sqrt{7+2\sqrt{10}}$.

Solution:

1) $7-\sqrt{3}-\sqrt{(7+\sqrt{3})^2}=7-\sqrt{3}-7-\sqrt{3}=-2\sqrt{3}$;

2) $\sqrt{(5-\sqrt{5})^2}+\sqrt{(5+\sqrt{5})^2}=5-\sqrt{5}+5+\sqrt{5}=10$;

3) $\sqrt{4-2\sqrt{3}}=\sqrt{3-2\sqrt{3}+1}=\sqrt{(\sqrt{3}-1)^2}=\sqrt{3}-1$;

4) $\sqrt{7+2\sqrt{10}}=\sqrt{5+2\sqrt{5}\sqrt{2}+2}=\sqrt{(\sqrt{5}+\sqrt{2})^2}=\sqrt{5}+\sqrt{2}$.

Rationalizing the denominator

To free the fraction $\frac{a}{\sqrt{c}}$ from irrationality in the denominator, the numerator and denominator of this fraction must be multiplied by $\sqrt{c}$, namely:

$$\frac{a}{\sqrt{c}}=\frac{a\cdot\sqrt{c}}{\sqrt{c}\cdot\sqrt{c}}=\frac{a\sqrt{c}}{c}.$$

The expressions $\left(\sqrt{a}+\sqrt{b}\right)$ and $\left(\sqrt{a}-\sqrt{b}\right)$ are called mutually conjugate, and their product is equal to

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=a-b \;\; (a\geq 0, b\geq 0).$$

Example 2.21. Rationalize the denominator:

$$a)\ \frac{2}{\sqrt{3}+\sqrt{2}};\;\; b)\ \frac{4}{\sqrt{7}-\sqrt{3}};\;\; c)\ \frac{ab}{\sqrt{a}+\sqrt{b}}.$$

Solution:

$$a)\ \frac{2}{\sqrt{3}+\sqrt{2}}=\frac{2\left(\sqrt{3}-\sqrt{2}\right)}{(\sqrt{3}+\sqrt{2})\left(\sqrt{3}-\sqrt{2}\right)}=\frac{2\left(\sqrt{3}-\sqrt{2}\right)}{3-2}=2\left(\sqrt{3}-\sqrt{2}\right);$$

$$b)\ \frac{4}{\sqrt{7}-\sqrt{3}}=\frac{4\left(\sqrt{7}+\sqrt{3}\right)}{\left(\sqrt{7}-\sqrt{3}\right)\left(\sqrt{7}+\sqrt{3}\right)}=\frac{4\left(\sqrt{7}+\sqrt{3}\right)}{7-3}=\sqrt{7}+\sqrt{3};$$

$$c)\ \frac{ab}{\sqrt{a}+\sqrt{b}}=\frac{ab\left(\sqrt{a}-\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)}=\frac{ab\left(\sqrt{a}-\sqrt{b}\right)}{a-b}.$$

The power of a real number with a rational index

If $m\in Z, n\in N, n\geq 2,\ a>0$, then $a^{\frac{m}{n}}=\sqrt[n]{a^m}$.

Comment. If $n=2k+1\ (k\in N)$, the formula $a^{\frac{m}{n}}=\sqrt[n]{a^m}$ is valid when $a\in R\setminus\{0\}$.

When converting irrational expressions, the properties of radicals and powers with a rational index are used.

Example 2.22. Simplify:

$$\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}-6}-\frac{3}{x^{\frac{1}{2}}+6}+\frac{x}{36-x}.$$

Solution: It's obvious that $x \geq 0,\ x \neq 36$. To simplify the expression, it is necessary to bring it to a common denominator using the formula $a^2 - b^2 = (a-b)(a+b)$, namely:

$$\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}-6} - \frac{3}{x^{\frac{1}{2}}+6} + \frac{x}{36-x} = \frac{x^{\frac{1}{2}}\left(x^{\frac{1}{2}}+6\right) - 3\left(x^{\frac{1}{2}}-6\right) - x}{x-36} =$$

$$= \frac{x + 6x^{\frac{1}{2}} - 3x^{\frac{1}{2}} + 18 - x}{x-36} = \frac{3x^{\frac{1}{2}}+18}{x-36} = \frac{3\left(x^{\frac{1}{2}}+6\right)}{x \quad -36} = \frac{3}{x^{\frac{1}{2}}-6}.$$

Example 2.23. Simplify:

$$\frac{\left(a^{1,5}+b^{1,5}\right):\left(a^{0,5}+b^{0,5}\right) - a^{0,5}b^{0,5}}{a-b} + \frac{2b^{0,5}}{a^{0,5}+b^{0,5}}.$$

Solution: It's obvious that $a \geq 0, b \geq 0, a \neq b$. To simplify the expression, you must do the division using the formula

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right),\ (a-b)^2 = a^2 - 2ab + b^2$$

and bring the fraction to a common denominator, namely:

$$\frac{\left(a^{1,5}+b^{1,5}\right):\left(a^{0,5}+b^{0,5}\right) - a^{0,5}b^{0,5}}{a-b} + \frac{2b^{0,5}}{a^{0,5}+b^{0,5}} =$$

$$= \frac{\frac{\left(a^{0,5}+b^{0,5}\right)\left(a - a^{0,5}b^{0,5} + b\right)}{a^{0,5}+b^{0,5}} - a^{0,5}b^{0,5}}{a-b} + \frac{2b^{0,5}}{a^{0,5}+b^{0,5}} =$$

$$= \frac{a - 2a^{0,5}b^{0,5} + b}{a-b} + \frac{2b^{0,5}}{a^{0,5}+b^{0,5}} = \frac{\left(a^{0,5}-b^{0,5}\right)^2}{\left(a^{0,5}-b^{0,5}\right)\left(a^{0,5}+b^{0,5}\right)} + \frac{2b^{0,5}}{a^{0,5}+b^{0,5}} =$$

$$= \frac{a^{0,5} - b^{0,5} + 2b^{0,5}}{a^{0,5}+b^{0,5}} = 1.$$

Example 2.24. Simplify:

$$A=\frac{1}{4}\left(xa^{-1}-ax^{-1}\right)\cdot\left(\frac{a^{-1}-x^{-1}}{a^{-1}+x^{-1}}-\frac{a^{-1}+x^{-1}}{a^{-1}-x^{-1}}\right).$$

Solution: It is obvious that $a\neq 0,\ x\neq 0,\ x\neq a,\ x\neq -a$.

Using $a^{-m}=\frac{1}{a^m}$, we have:

$$A=\frac{1}{4}\left(\frac{x}{a}-\frac{a}{x}\right)\cdot\left(\frac{x-a}{x+a}-\frac{x+a}{x-a}\right)=$$

$$=\frac{1}{4}\cdot\frac{x^2-a^2}{ax}\cdot\frac{x^2-2ax+a^2-x^2-2ax-a^2}{x^2-a^2}=\frac{1}{4}\cdot(-4)=-1 .$$

Example 2.25. Simplify:

$$A=\frac{y^2\left(xy^{-1}-1\right)^2}{x\left(1+x^{-1}y\right)^2}\cdot\frac{y^2\left(x^{-2}+y^{-2}\right)}{x\left(xy^{-1}-x^{-1}y\right)}:\frac{1-x^{-1}y}{xy^{-1}+1}.$$

Solution: It is obvious that $x\neq 0, y\neq 0, x\neq -y, x\neq y$.

Using $a^{-m}=\frac{1}{a^m}$ and reduction formulas we have:

$$A=\frac{y^2(x-y)^2\cdot x^2}{y^2x(x+y)^2}\cdot\frac{y^2\left(y^2+x^2\right)xy}{x^2y^2x\left(x^2-y^2\right)}\cdot\frac{(x+y)x}{y(x-y)}=\frac{x^2+y^2}{(x+y)^2}.$$

Example 2.26. Simplify:

$$\frac{(a-b)^3\left(\sqrt{a}+\sqrt{b}\right)^{-3}+2a\sqrt{a}+b\sqrt{b}}{a\sqrt{a}+b\sqrt{b}}+\frac{3\left(\sqrt{ab}-b\right)}{a-b}.$$

Solution: It is obvious that $a \geq 0, b \geq 0, a \neq b$. To simplify the expression, you must use the formulas

$$a^{-m} = \frac{1}{a^m},\ (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2),\ \ a^2 - b^2 = (a-b)(a+b),$$

the rule of taking the factor out from under the root and putting it under the root, and reduction of fractions to a common denominator. So, we have:

$$\frac{(a-b)^3(\sqrt{a}+\sqrt{b})^{-3} + 2a\sqrt{a} + b\sqrt{b}}{a\sqrt{a} + b\sqrt{b}} + \frac{3(\sqrt{ab} - b)}{a-b} =$$

$$= \frac{\left(\dfrac{a-b}{\sqrt{a}+\sqrt{b}}\right)^3 + 2a\sqrt{a} + b\sqrt{b}}{a\sqrt{a} + b\sqrt{b}} + \frac{3\sqrt{b}(\sqrt{a} - \sqrt{b})}{a-b} =$$

$$= \frac{(\sqrt{a}-\sqrt{b})^3 + 2a\sqrt{a} + b\sqrt{b}}{a\sqrt{a} + b\sqrt{b}} + \frac{3\sqrt{b}}{\sqrt{a}+\sqrt{b}} =$$

$$= \frac{\sqrt{a^3} - 3a\sqrt{b} + 3\sqrt{a}b - \sqrt{b^3} + 2\sqrt{a^3} + \sqrt{b^3}}{\sqrt{a^3} + \sqrt{b^3}} + \frac{3\sqrt{b}}{\sqrt{a}+\sqrt{b}} =$$

$$= \frac{3\sqrt{a}\ (a - \sqrt{ab} + b)}{(\sqrt{a}+\sqrt{b})(a - \sqrt{ab} + b)} + \frac{3\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{3(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}} = 3.$$

Example 2.27. Simplify:

$$\left(\frac{\sqrt[4]{a^3} - b}{\sqrt[4]{a} - \sqrt[3]{b}} - 3 \cdot \sqrt[12]{a^3b^4}\right)^{-\frac{1}{2}} \cdot \left(\frac{\sqrt[4]{a^3} + b}{\sqrt[4]{a} + \sqrt[3]{b}} - \sqrt[3]{b^2}\right).$$

Solution: It is obvious that $a \geq 0,\ \sqrt[4]{a} \neq \sqrt[3]{b}$. To simplify the expression, use the formulas: $a^{-m} = \dfrac{1}{a^m}$, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$,

$(a-b)^2=a^2-2ab+b^2$, root properties, reduction of fractions to a common denominator, namely:

$$\left(\frac{\sqrt[4]{a^3}-b}{\sqrt[4]{a}-\sqrt[3]{b}}-3\cdot\sqrt[12]{a^3b^4}\right)^{-\frac{1}{2}}\cdot\left(\frac{\sqrt[4]{a^3}+b}{\sqrt[4]{a}+\sqrt[3]{b}}-\sqrt[3]{b^2}\right)=$$

$$\cdot\left(\frac{\left(\sqrt[4]{a}+\sqrt[3]{b}\right)\left(\sqrt[4]{a^2}-\sqrt[4]{a}\sqrt[3]{b}+\sqrt[3]{b^2}\right)}{\sqrt[4]{a}+\sqrt[3]{b}}-\sqrt[3]{b^2}\right)^{-1/2}=$$

$$=\left(\sqrt[4]{a^2}-2\sqrt[4]{a}\sqrt[3]{b}+\sqrt[3]{b^2}\right)^{-\frac{1}{2}}\cdot\left(\sqrt[4]{a^2}-\sqrt[4]{a}\sqrt[3]{b}\right)=$$

$$=\frac{1}{\sqrt{\left(\sqrt[4]{a}-\sqrt[3]{b}\right)^2}}\cdot\sqrt[4]{a}\left(\sqrt[4]{a}-\sqrt[3]{b}\right)=\frac{\sqrt[4]{a}\left(\sqrt[4]{a}-\sqrt[3]{b}\right)}{\left|\sqrt[4]{a}-\sqrt[3]{b}\right|}=\left[\begin{array}{l}\sqrt[4]{a},\ if\ \sqrt[4]{a}>\sqrt[3]{b};\\-\sqrt[4]{a},\ if\ \sqrt[4]{a}<\sqrt[3]{b}.\end{array}\right.$$

Example 2.28. Simplify:

$$\frac{\sqrt[3]{a+\sqrt{2-a^2}}\cdot\sqrt[6]{1-a\sqrt{2-a^2}}}{\sqrt[3]{1-a^2}}.$$

Solution: Using $2-a^2\geq 0$, we have $|a|\leq\sqrt{2}$, and the properties of powers, and the following formula:

$$(a+b)^2=a^2+2ab+b^2,\ \ a^2-b^2=(a-b)(a+b),$$

we have:

$$\frac{\sqrt[3]{a+\sqrt{2-a^2}}\cdot\sqrt[6]{1-a\sqrt{2-a^2}}}{\sqrt[3]{1-a^2}}=\frac{\sqrt[6]{\left(a+\sqrt{2-a^2}\right)^2}\cdot\sqrt[6]{1-a\sqrt{2-a^2}}}{\sqrt[3]{1-a^2}}=$$

$$=\frac{\sqrt[6]{2\left(1+a\sqrt{2-a^2}\right)\left(1-a\sqrt{2-a^2}\right)}}{\sqrt[3]{1-a^2}}=\frac{\sqrt[6]{2\left(1-a^2\right)^2}}{\sqrt[3]{1-a^2}}=\frac{\sqrt[6]{2}\cdot\sqrt[3]{\left|1-a^2\right|}}{\sqrt[3]{1-a^2}}=$$

$$=\left[\begin{array}{l}\sqrt[6]{2},\quad if\quad 1-a^2>0,\\-\sqrt[6]{2},if\quad 1-a^2<0.\end{array}\right.=\left[\begin{array}{l}\sqrt[6]{2},\ a^2<1,\\-\sqrt[6]{2},1<a^2\leq 2.\end{array}\right.=\left[\begin{array}{l}\sqrt[6]{2},\ |a|<1,\\-\sqrt[6]{2},\ 1<|a|\leq\sqrt{2}.\end{array}\right.$$

Example 2.29. Simplify:

$$A=\sqrt[4]{\left(1-2a+a^2\right)\left(a^2-1\right)\left(a-1\right)}:\frac{a^2+2a-3}{\sqrt[4]{a+1}}.$$

Solution: Use the obvious constraint: $a+1>0$. As a result, we obtain:

$$A=\sqrt[4]{(a-1)^4\,(a+1)}:\frac{(a-1)(a+3)}{\sqrt[4]{a+1}}=\frac{|a-1|\sqrt[4]{a+1}\cdot\sqrt[4]{a+1}}{(a-1)(a+3)}=$$
$$=\frac{|a-1|\sqrt[4]{(a+1)^2}}{(a-1)(a+3)}=\frac{|a-1|\sqrt{a+1}}{(a-1)(a+3)}.$$

Consider two cases $a-1>0$ and $a-1<0$ taking into account the domain $a>-1$.

1. If $-1<a<1$, then $|a-1|=1-a$; and we obtain:

$$A=\frac{-(a-1)\sqrt{a+1}}{(a-1)(a+3)}=\frac{-\sqrt{a+1}}{a+3}.$$

2. If $a>1$, then $|a-1|=a-1$ and we obtain:

$$A=\frac{(a-1)\sqrt{a+1}}{(a-1)(a+3)}=\frac{\sqrt{a+1}}{a+3}.$$

When solving examples containing several modules, the interval method is used, the essence of which is as follows:

a) all values of a variable are applied to the domain of a given expression, where the expressions under the modulus sign are zeroed, and with which the axis is divided into a number of intervals;

b) in each of the obtained intervals, the given expression is simplified.

Example 2.30. Simplify:

$$A=\left(\frac{|x-1|}{x-1}\cdot x^2-2x\cdot\frac{|x+1|}{x+1}+2x-4\right):|x-2|.$$

Solution: We apply the above method to simplify this expression. It is obvious that: $x \neq \pm 1$ and $x \neq 2$.

Points $x=-1, x=1, x=2$, at which the modules vanish, divide the numerical axis into a series of intervals (Fig.).

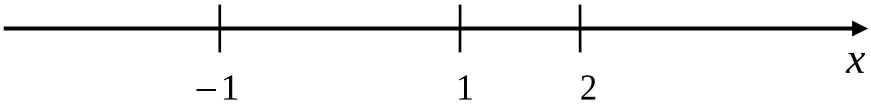


Fig. **The numerical axis**

a) if $x<-1$, then $|x-1|=-(x-1)$, $|x+1|=-(x+1)$, $|x-2|=-(x-2)$, then:

$$A=\left(\frac{-(x-1)}{x-1}\cdot x^2+2x\cdot\frac{x+1}{x+1}+2x-4\right):(2-x)=\frac{x^2-4x+4}{x-2}=x-2;$$

b) if $-1<x<1$, then $|x-1|=-(x-1)$, $|x+1|=x+1$, $|x-2|=2-x$; then:

$$A=\left(\frac{-(x-1)}{x-1}\cdot x^2-2x\cdot\frac{x+1}{x+1}+2x-4\right):(2-x)=\frac{x^2+4}{x-2};$$

c) if $1<x<2$, then $|x-1|=x-1$, $|x+1|=x+1$, $|x-2|=2-x$; then:

$$A=\left(\frac{x-1}{x-1}\cdot x^2-2x\cdot\frac{x+1}{x+1}+2x-4\right):(2-x)=\frac{x^2-4}{2-x}=-x-2;$$

d) if $x>2$, then $|x-1|=x-1$, $|x+1|=x+1$, $|x-2|=x-2$; then:

$$A=\left(\frac{x-1}{x-1}\cdot x^2-2x\cdot\frac{x+1}{x+1}+2x-4\right):(x-2)=\frac{x^2-4}{x-2}=x+2.$$

Answer: if $x<-1$, $A=x-2$; if $-1<x<1$, $A=\dfrac{x^2+4}{x-2}$;

if $1<x<2$, $A=-x-2$; if $x>2$, $A=x+2$.

Tasks for individual work

2.31. Perform the operations:

1) $(a+x)^2$; 2) $\left(\dfrac{t}{3}-\dfrac{2}{t}\right)^2$; 3) $\left(\dfrac{1}{4}a-2b\right)^2$; 4) $\left(2a^k-3a^{2-k}\right)^2$;

5) $(1-3a)(1+3a)(1+9a^2)$; 6) $\left(3^{2m}+3^{3n}\right)\left(3^{2m}-3^{3n}\right)\left(3^{4m}-3^{6n}\right)$;

7) $\left(\frac{1}{8}a^3+\frac{1}{27}b^3\right):\left(\frac{1}{2}a+\frac{1}{3}b\right)$.

2.32. Bring the common multiplier out of parentheses:

1) $5y^3-15y^2$; 2) $a^{12}-a^2b$; 3) $8^{2n-1}+8^{2n+1}$; 4) $2\cdot 3^{x+1}-4\cdot 3^{x-1}+3^x$; 5) $5a(2b-3)-3b(2b-3)+3-2b$; 6) $6a^2b(2x^2-3y^2)-9ab(2x^2-3y^2)$.

2.33. Expand the polynomials into factors:

1) $ac+bc+a+b$; 2) $ax+ay-x-y$; 3) $15b^2c-48ab^2-10c^2+32ac$;
4) $a^3+a^2b-ab-b^2$; 5) $(x^2+4x+8)^2-3x(x^2+4x+8)+2x^2$.

2.34. Simplify:

$$\left(\frac{2a+10}{3a-1}+\frac{130-a}{1-3a}+\frac{30}{a}-3\right)\cdot\frac{3a^3+8a^2-3a}{1-\frac{1}{4}a^2}.$$

2.35. Simplify and calculate:

1) $\frac{x}{x^2+y^2}-\frac{y(x-y)^2}{x^4-y^4}$ for $x=1$ and $y=4$;

2) $\left(\frac{a^2}{a+b}-\frac{a^3}{a^2+2ab+b^2}\right):\left(\frac{a}{a+b}-\frac{a^2}{a^2-b^2}\right)$ for $a=-2{,}5;\ b=0{,}5$.

2.36. Simplify:

$$\left(\frac{3(x+2)}{2(x^3+x^2+x+1)}+\frac{2x^2-x-10}{2(x^3-x^2+x-1)}\right):\left(\frac{5}{x^2+1}+\frac{3}{2(x+1)}-\frac{3}{2(x-1)}\right).$$

2.37. Simplify:

$$\left(\frac{x-y}{2y-x}-\frac{x^2+y^2+y-2}{x^2-xy-2y^2}\right):\frac{4x^4+4x^2y+y^2-4}{x^2+y+xy+x}.$$

2.38. Simplify:

$$(a+b)^{-1}\cdot\frac{a^{-2}+b^{-2}}{a^{-1}+b^{-1}}:\left(\frac{ab}{a^2+b^2}\right)^{-1}\cdot\left(\frac{2ab}{a+b}\right)^{-1}.$$

2.39. Simplify:

$$\left(\frac{1+ax^{-1}}{a^{-1}x^{-1}}\cdot\frac{a^{-1}-x^{-1}}{a^{-1}x-ax^{-1}}\right):\frac{ax^{-1}}{x-a}.$$

2.40. Simplify:

$$\frac{\left(ab^{-1}+a^{-1}b+1\right)\left(a^{-1}-b^{-1}\right)^2}{a^2b^{-2}+a^{-2}b^2-\left(ab^{-1}+a^{-1}b\right)}.$$

2.41. Calculate:

1) $7-\sqrt{3}-\sqrt{\left(7+\sqrt{3}\right)^2}$; 2) $\sqrt{5-2\sqrt{6}}$; 3) $\sqrt{7-4\sqrt{3}}$.

2.42. Simplify:

$$\left(\frac{\sqrt{x}+2}{\sqrt{x}-2}+\frac{\sqrt{x}-2}{\sqrt{x}+2}-\frac{16}{x-4}\right)^{-2}.$$

2.43. Simplify:

$$\left(\frac{a\sqrt{a}+b\sqrt{b}}{\sqrt{a}+\sqrt{b}}-\sqrt{ab}\right):(a-b)+\frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}}.$$

2.44. Simplify:

$$\left(\frac{\sqrt{a}}{2}-\frac{1}{2\sqrt{a}}\right)^2\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}-\frac{\sqrt{a}+1}{\sqrt{a}-1}\right).$$

2.45. Simplify: $\dfrac{\left(a^{-1}+b^{-1}\right)(a+b)^{-1}}{\sqrt[6]{a^4\sqrt[5]{a^{-2}}}}$.

Questions for self-assessment

1. Transformation of rational expressions.
2. The power of a real number with a natural exponent.
3. What properties of exponents do you know?
4. The addition and product of polynomials.
5. What is the rationalization of the denominator?
6. The power of a real number with a rational index.

3. Functions and graphics

3.1. The concept of a function. Ways to set functions. The monotony of a function

The concept of a function

Let two variables x and y be given with their domains $D(f)$ and $E(f)$. The variable y is called a function of the variable x if, according to a certain rule or law, only one specific value $y \in E(f)$ is assigned to each value $x \in D(f)$.

This correspondence is commonly written as: $y = f(x)$. $D(f)$ is called the function domain, and $E(f)$ is the function codomain, x is the argument (independent variable), y is the function.

Example 3.1. Find the domain:

1) $y = \dfrac{2x-1}{3x-6}$; 2) $y = \dfrac{5x+1}{x^2+2x+5}$; 3) $y = \sqrt{2x-1}$;

4) $y = \sqrt{x+2} + \sqrt{x-1}$; 5) $y = log_2\left(x^2 - 2x - 8\right)$; 6) $y = \sqrt{\dfrac{1-x}{2x-4}}$.

Solution:

1. $y = \dfrac{2x-1}{3x-6}$. This function is defined for all values of x, except those where the denominator of the fraction is zero. Solving the equation $3x - 6 = 0$, we find: $x = 2$. Therefore, the domain of this function is: $R \setminus \{2\}$.

2. $y=\frac{5x+1}{x^2+2x+5}$. This fraction exists for all values except those for which $x^2+2x+5=0$. Find the roots of this equation:

$$x=-1\pm\sqrt{1-5};\quad x=-1\pm\sqrt{-4}.$$

The equation has no real roots. Therefore, this function is defined for any x.

3. $y=\sqrt{2x-1}$. The square root is defined only for non-negative numbers. Therefore, the domain of definition of this function is considered as a set satisfying the inequality:

$$2x-1\geq 0\text{, then } x\geq\frac{1}{2}.$$

4. $y=\sqrt{x+2}+\sqrt{x-1}$. Given that square roots are defined only for non-negative numbers, we have a system of inequalities:

$$\begin{cases} x+2\geq 0; \\ x-1\geq 0. \end{cases} \Rightarrow \begin{cases} x\geq -2; \\ x\geq 1. \end{cases} \Rightarrow \quad x\geq 1.$$

Therefore, the domain of this function is: $[1;+\infty)$.

5. $y=log_2(x^2-2x-8)$. Logarithms are defined only for positive numbers. Therefore, the domain of this function is defined as the set of values which satisfy the inequality:

$$x^2-2x-8>0.$$

Find the roots of the quadratic polynomials x^2-2x-8. To do this, we solve the equation $x^2-2x-8=0$: $x=1\pm\sqrt{1+8}$; $x_1=-2$; $x_2=4$. Then the inequality can be rewritten as:

$$(x+2)(x-4)>0.$$

To solve this inequality, we use the interval method, noting on the number line the signs: "+" or "−" (Fig. 3.1).

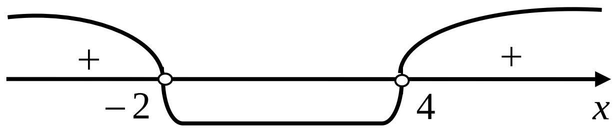


Fig. 3.1. **Plotting the domain of the function $y = log_2\,(x^2 - 2x - 8)$ on the number axis**

So, the domain of this function is the union of the intervals: $x \in (-\infty;-2) \cup (4;+\infty)$.

6. $y = \sqrt{\dfrac{1-x}{2x-4}}$. The square root is defined only for non-negative numbers. Therefore, the domain of definition of this function is the set of all values x which satisfy the inequality: $\dfrac{1-x}{2x-4} \geq 0.$

To solve the inequality, we use the interval method, marking the inequality sign on the number line (Fig. 3.2).

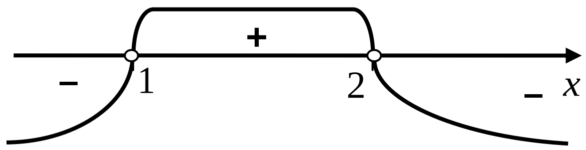


Fig. 3.2. **Plotting the domain of the function $y = \sqrt{\dfrac{1-x}{2x-4}}$ on the number axis**

Given that when $x = 2$, the denominator vanishes, the domain of the definition of this function is the set of all real numbers satisfying the inequality: $-1 \leq x < 2$.

Ways to set functions

1. The analytical way of setting a function. In this method, the function is set using the formula, for example,

$$y = 2x - 1; \quad y = x^2; \quad y = \sqrt{x+1}; \quad y = \begin{cases} 2x^2, & \text{if} \quad x \leq 0 \\ x, & \text{if} \quad x > 0 \end{cases}.$$

2. The graphic way to set a function. With this method in hand we depict in the XOY coordinate system a graph of the function $y=f(x)$.

The graph of the function $y=f(x)$ is called a set of all points of the coordinate plane with coordinates $(x, f(x))$, where the abscissa x is the value of the argument, and the ordinate $y=f(x)$ is the corresponding value of the function (Fig. 3.3).

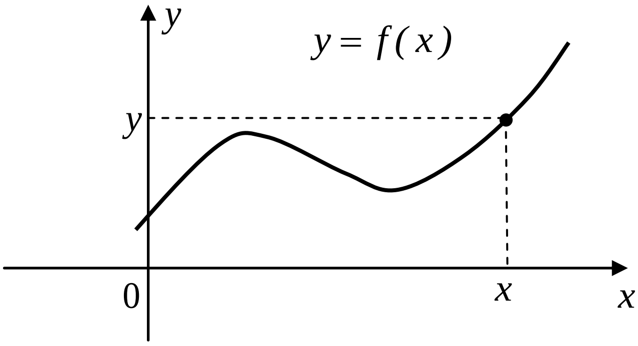


Fig. 3.3. **A function graph**

3. The tabular way of setting a function. With this method, the values of the argument $x_1, x_2, ..., x_n$ and the corresponding values of the function $y_1, y_2, ..., y_n$ are written out in a certain order (Fig. 3.4).

x	x_1	x_2	$\dots$	x_n
y	y_1	y_2	$\dots$	y_n

Fig. 3.4. **The tabular way of setting a function**

Monotonic functions

A function $y=f(x)$ is called increasing on a given numerical interval, if the larger value of the argument from this set gives the larger value of the function, that is, if $x_2 > x_1$, then $f(x_2) > f(x_1)$. For example, $y=3x$ is the increasing function.

A function $y=f(x)$ is called a decreasing one on a given numeric interval if the larger value of the argument from this set corresponds to the smaller value of the function, that is, if $x_2 > x_1$, then $f(x_2) < f(x_1)$. For example, $y=-x^3$ is the decreasing function.

A monotonic function is a function which is either entirely nonincreasing or nondecreasing.

Even and odd functions

A function is called even if for any x and $-x$ from the domain of the function the following equality is true: $f(-x)=f(x)$. For example, the function $y=x^4+1$ is even because

$$f(-x)=(-x)^4+1=x^4+1=f(x).$$

A function is called odd if for any x and $-x$ from the function' domain the equality $f(-x)=-f(x)$ is fulfilled. For example, the function $y=x^3+x$ is odd because $f(-x)=(-x)^3-x=-(x^3+x)=-f(x)$.

If neither $f(-x)\neq f(x)$ nor $f(-x)\neq -f(x)$ are fulfilled, the function $f(x)$ is called neither odd nor even.

It is possible to say that the function is even or odd, only in the case when the domain is symmetric to the origin of coordinates.

For example, the function $y=lg\,x$ is defined only for $x>0$. Because of that the term $lg(-x)$ has no sense. Therefore, saying whether the function $y=lg\,x$ is even or odd does not make sense either. It should be noted that the graph of the even function is symmetric to the ordinate axis (Fig. 3.5), and the odd function is relative to the origin of the coordinates (Fig. 3.6).

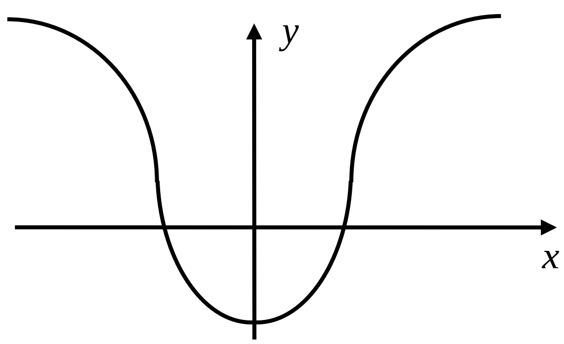


Fig. 3.5. **The even function**

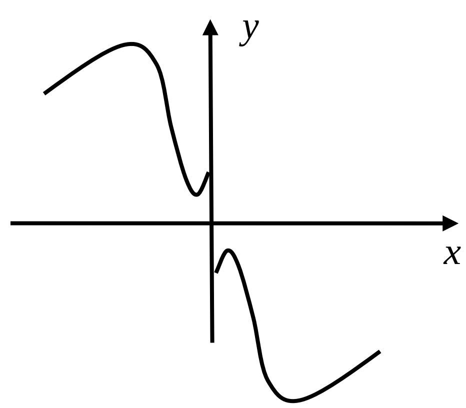


Fig. 3.6. **The odd function**

Example 3.2. Find out which functions are even, odd or functions of the general form (neither odd nor even).

1) $f(x)=2x^3-x$; 2) $f(x)=x+tg\,x$ 3) $f(x)=x^2+4x^4$;

4) $f(x)=cos\,x+x^3$; 5) $f(x)=(x+2)^2$; 6) $f(x)=\sqrt{x^2+1}$.

Solution: To study the function for evenness or oddness, consider $f(-x)$:

1) $f(x)=2x^3-x$;

$$f(-x)=2(-x)^3-(-x)=-2x^3+x=-(2x^3-x)=-f(x).$$

Because $f(-x)=-f(x)$, this function is odd;

2) $f(x)=x+tg\,x$;

$$f(-x)=(-x)+tg(-x)=-x-tgx=-(x+tgx)=-f(x).$$

This function is odd because $f(-x)=-f(x)$;

3) $f(x)=x^2+4x^4$;

$$f(-x)=(-x)^2+4(-x)^4=x^2+4x^4=f(x).$$

Because $f(-x)=f(x)$, we can make an inference: the function is even;

4) $f(x)=\cos x+x^3$;

$$f(-x)=\cos(-x)+(-x)^3=\cos x-x^3\neq f(x).$$

This function is of a general form;

5) $f(x)=(x+2)^2$;

$$f(-x)=(-x+2)^2\neq\pm f(x).$$

The function is of a general form;

6) $f(x)=\sqrt{x^2+1}$;

$$f(-x)=\sqrt{(-x)^2+1}=\sqrt{x^2+1}=f(x).$$

Because $f(-x)=f(x)$, the function is even.

7) $f(x)=\sin 5x\cdot\cos 7x$;

$$f(-x) = sin(5 \cdot (-x)) \cdot cos(7 \cdot (-x)) = -sin(5x) \cdot cos(7x) = -f(x).$$

Because $f(-x) = -f(x)$, the function is odd.

3.2. Elementary functions and their graphics

The linear function ($y = kx + b$)

Here, k and b are some numbers. The function $y = kx + b$ is one of the most common elementary functions.

The domain of this function is a set of all real numbers.

The function $y = kx + b$ is neither even nor odd.

The graph of this function is a straight line, for the construction of which you need to have two points (the easiest way is to take two points on the coordinate axis: $x = 0$, $y = b; y = 0$, $x = -\frac{b}{k}$.

Depending on the sign of k (the slope) the straight line forms an acute or obtuse angle with the abscissa axis (Fig. 3.7) and (Fig. 3.8).

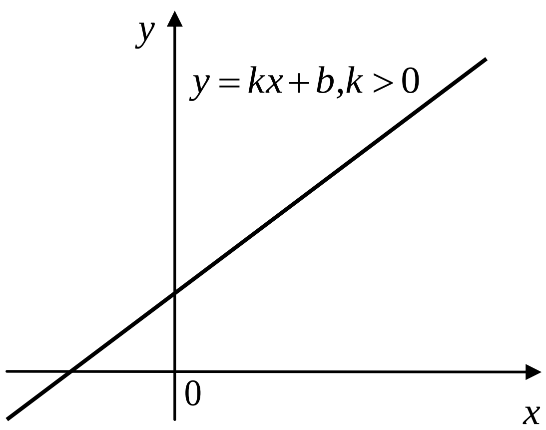


Fig. 3.7. **The graph of the function** $y = kx + b$, $k > 0$

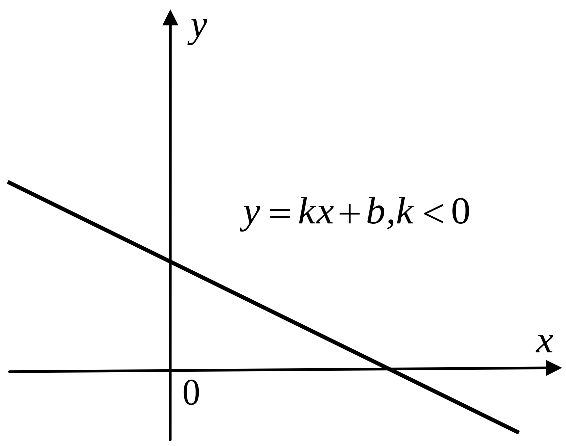


Fig. 3.8. **The graph of the function** $y = kx + b$, $k < 0$

For $k > 0$ the function is increasing and for $k < 0$ the function is decreasing.

If $k = 0$, the function is reduced to $y = b$. The graph of this function is a straight line parallel to the x-axis (Fig. 3.9).

If $k=0$ and $b=0$, the function is reduced to $y=0$. The graph of this function is a straight line, which coincides with the x-axis.

For $k \neq 0$ and $b=0$ the function is reduced to $y=kx$. The graph of this function is a straight line which passes through the origin of the coordinates (Fig. 3.10).

If $k=1$, the line $y=x$ is called the bisector of the 1st and 3rd coordinate angles.

If $k=-1$, the line $y=-x$ is called the bisector of the 2nd and 4th coordinate angles.

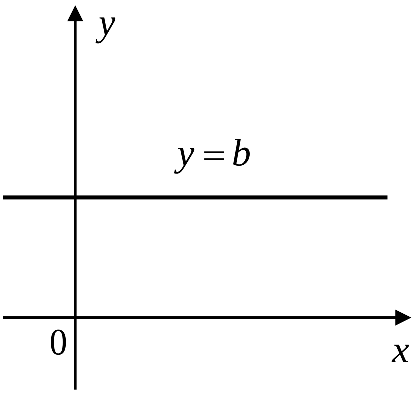


Fig. 3.9. **The graph of the function** $y=kx+b$, $k=0$

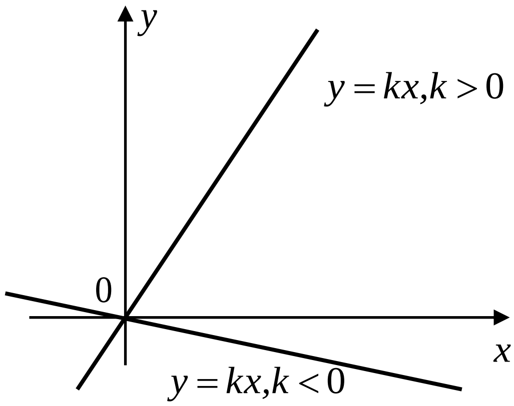


Fig. 3.10. **The graph of the function** $y=kx+b$, $b=0$

Example 3.3. Build the function graphs: a) $y=5x+4$; b) $y=-2x+4$; c) $y=2$.

Solution: a) $y=5x+4$;

To build a graph of the function, you need to select two points: $x=0$, then $y=4$; and $x=1$, then $y=9$. These calculations are conveniently arranged in Table 3.1.

Table 3.1

The values of *x, y*

x	0	1
y	4	9

We draw a straight line through the points $(0,4)$ and $(1,9)$ (Fig. 3.11).

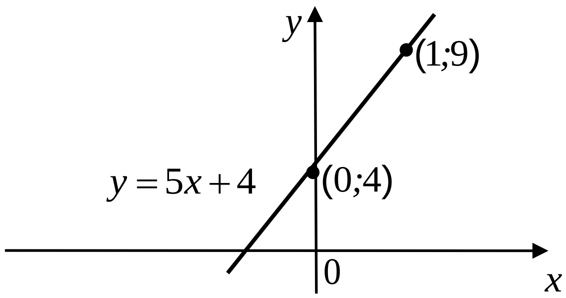


Fig. 3.11. **The strait line for** $y = 5x + 4$

b) $y = -2x + 4$;

To build a line, you can take any two points. Take the points on the coordinate axes, that is, first take $x = 0$ and find y, then take $y = 0$ and find x. Calculations are placed in Table 3.2:

Table 3.2

The values of *x, y*

x	0	2
y	4	0

We draw a straight line through the points $(0{,}4)$ and $(2{,}0)$ (Fig. 3.12).

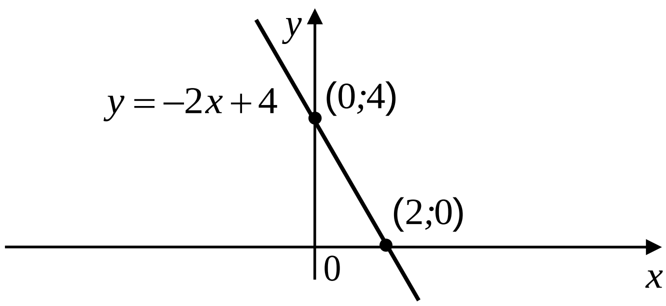


Fig. 3.12. **The straight line for** $y = -2x + 4$

c) $y = 2$;

The graph of this function is a straight line that is parallel to the x-axis and cuts a segment equal to 2 on the y-axis (Fig. 3.13).

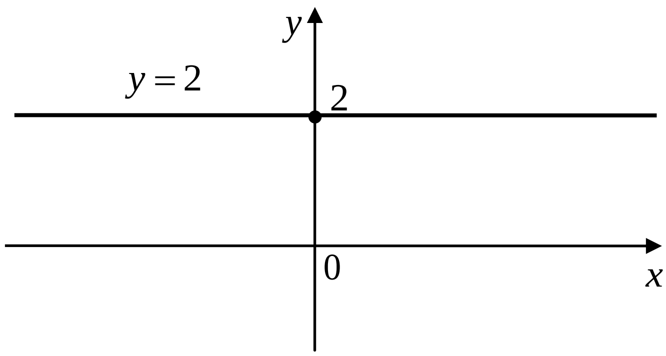


Fig. 3.13. **The horizontal line for** $y = 2$

Example 3.4. Determine the sign of k and b of $y = kx + b$, the graph of which is shown in Fig. 3.14.

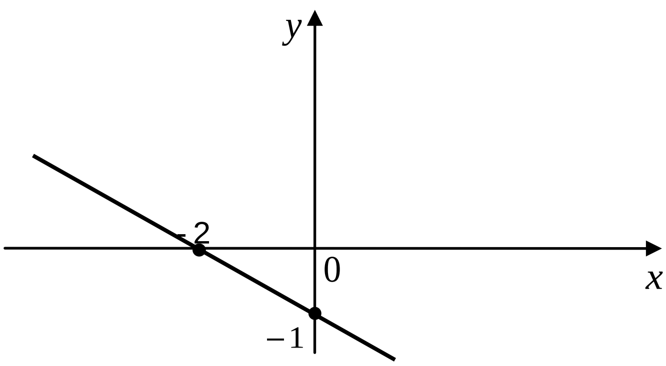


Fig. 3.14. **The straight line for** $y = kx + b$

Solution: It can be seen from the above that the straight line intersects the ordinate axis below zero, therefore $b < 0$. It is also clear that the function decreases and the straight line forms an obtuse angle with the x-axis, therefore $k < 0$.

The function $y = \frac{k}{x}$ $(k \neq 0)$

This is the function of an inverse proportionality, k is the coefficient of the inverse proportionality. The domain of this function is the set:

$$D(f) = (-\infty;0) \cup (0;+\infty) = R \setminus \{0\},$$

that is, the function graph does not cross the Oy-axis.

This function has no zeros, that is, the graph does not cross the Ox-axis.

The function $y = \frac{k}{x}$ is odd, that is, its graph is symmetrical relative to the origin (Fig. 3.15 and 3.16). The curve is called the hyperbola and consists of two branches.

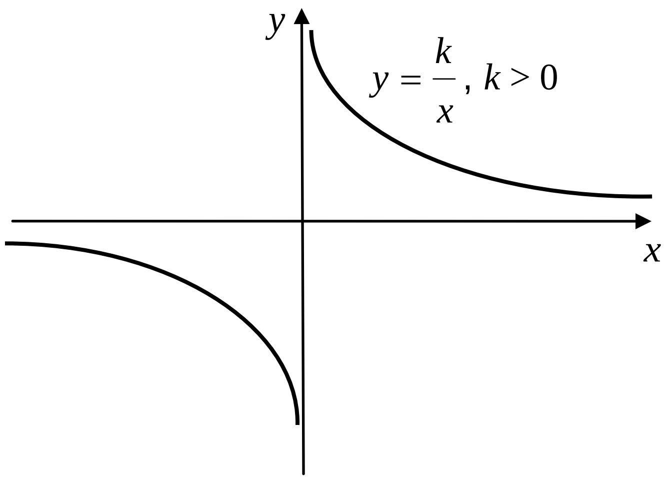


Fig. 3.15. **The hyperbola of the function** $y = \frac{k}{x}, k > 0$

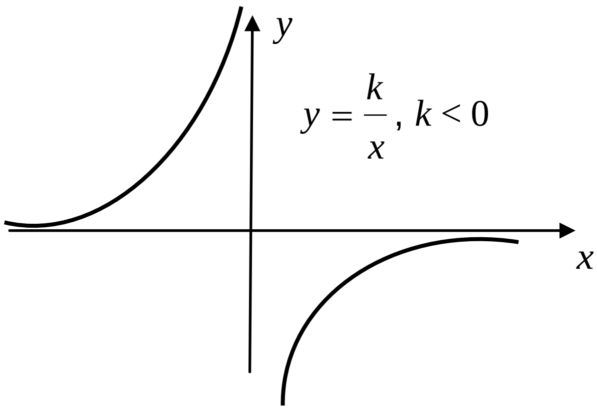


Fig. 3.16. **The hyperbola of the function** $y = \frac{k}{x}, k < 0$

Example 3.5. Build the function graphs: a) $y = \frac{2}{x}$; b) $y = -\frac{4}{x}$.

Solution: a) $y = \frac{2}{x}$. The graph of the function $y = \frac{k}{x}$ is a hyperbola. Because in the example $k > 0$, the branches of the hyperbola are located in the first and third quarters. For a more accurate construction, take a few points. The data for calculations are given in Table 3.3.

Table 3.3

The values of *x*, *y*

x	$1/2$	1	2	4
y	4	2	1	$1/2$

Considering that the function is odd and the graph lies in the third quarter, the construction will be performed using symmetry to the origin $O(0;0)$ (Fig. 3.17);

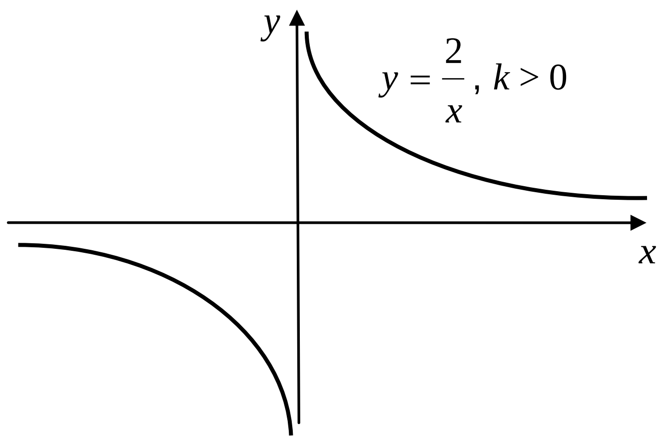


Fig. 3.17. **The hyperbola of** $y=\frac{2}{x}$, $k>0$

b) $y=-\frac{4}{x}$. The graph of this function is a hyperbola whose branches are located in the second and fourth quarters. For a more accurate construction, take a few points (Table 3.4).

Table 3.4

The points for the hyperbola

x	1	2	4	8
y	-4	-2	-1	$-1/2$

Given the oddness, the construction in the second quarter will be performed using symmetry with respect to the origin (Fig. 3.18):

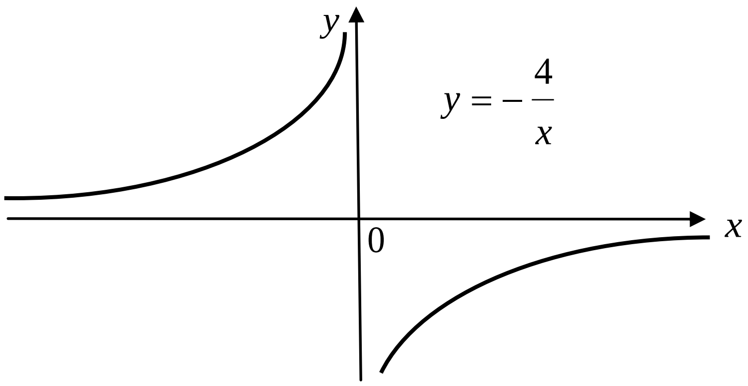


Fig. 3.18. **The hyperbola of the function** $y = -\frac{4}{x}$

The function $y = ax^2$ $(a \neq 0)$

The domain of this function is the set of all real numbers: $x \in R$.

This function is even, which means that the graph of this function is symmetric to the axis Oy.

The function equals zero for $x = 0$. For $a < 0$, $y < 0$, that is, the graph of the function lies below the axis Ox. For $x \in (-\infty;0)$ the function increases. For $x \in (0;+\infty)$ the function decreases.

For $a > 0$, $y > 0$, that is, the graph of the function lies above the axis Ox. For $x \in (-\infty;0)$ the function decreases. For $x \in (0;+\infty)$ the function increases.

The function graph $y = ax^2$ is called a parabola (Fig. 3.19 and 3.20).

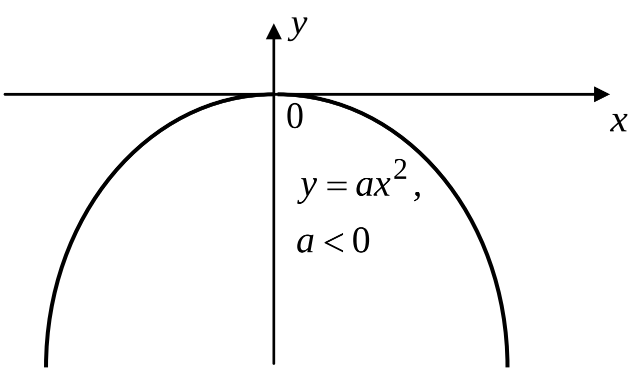


Fig. 3.19. **The parabola,** $a < 0$

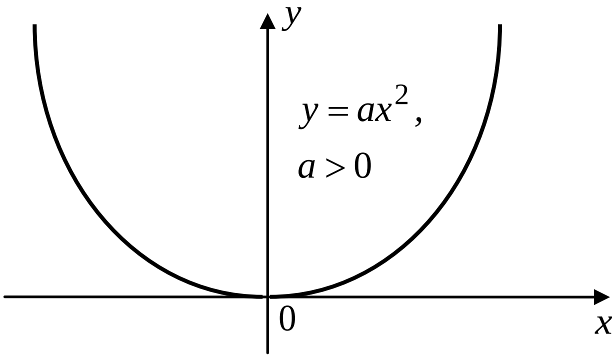


Fig. 3.20. **The parabola,** $a > 0$

The function $y = ax^2 + bx + c\,(a \neq 0)$

This function, where a,b,c are real numbers, is called quadratic. The domain of this function is $D(y) = R$. The graph of $y = ax^2$ is a parabola whose branches are directed up or down depending on the sign of the coefficient a. (For $a > 0$ it is up, for $a < 0$ it is down).

For $b = 0$ the function takes the form $y = ax^2 + c$ and is even and its graph is symmetric to the *y*-axis.

In general, $y = ax^2 + bx + c\,(a \neq 0)$ is neither even nor odd, that is, the graph does not have a symmetry to the origin and the *y*-axis.

To plot the function graph $y = ax^2 + bx + c\,(a \neq 0)$ it might be transformed into the form $y = a(x + x_0)^2 + y_0$ with the aid of completing the square as follows:

$$y = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) =$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Summing, $y = a(x + x_0)^2 + y_0$, where $x_0 = \frac{b}{2a}, y_0 = \frac{4ac - b^2}{4a} = -\frac{D}{4a}$, $D = b^2 - 4ac$ is the discriminant of $ax^2 + bx + c$. The point $(-x_0, y_0)$ is the top of the parabola, and the straight line $x = -x_0 = -\frac{b}{2a}$ is the axis of symmetry of the parabola.

If $D > 0$, then the parabola intersects the *x*-axis at two points; if $D < 0$, then there are no intersection points with the *x*-axis, if $D = 0$, the parabola touches the *x*-axis.

The parabola plots depending on a and D are given in Fig. 3.21 – 3.26.

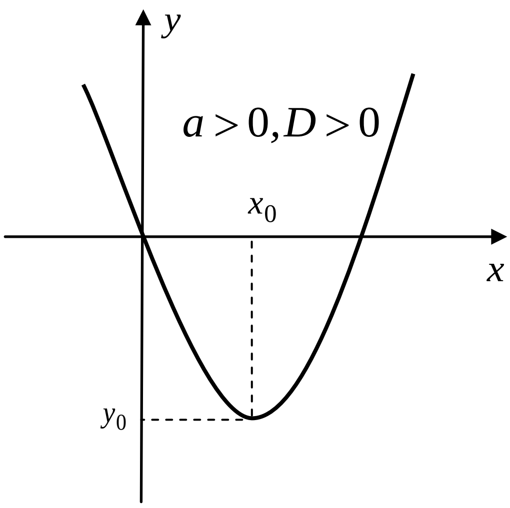


Fig. 3.21. **The parabola for** $a > 0$, $D > 0$

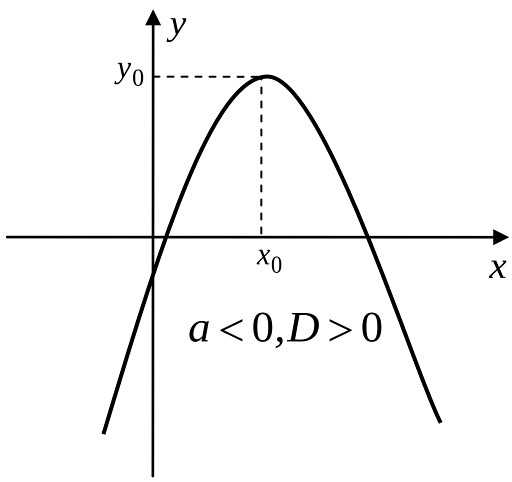


Fig. 3.22. **The parabola for** $a < 0$, $D > 0$

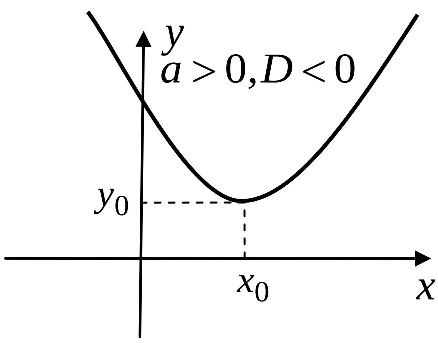


Fig. 3.23. **The parabola for** $a > 0$, $D < 0$

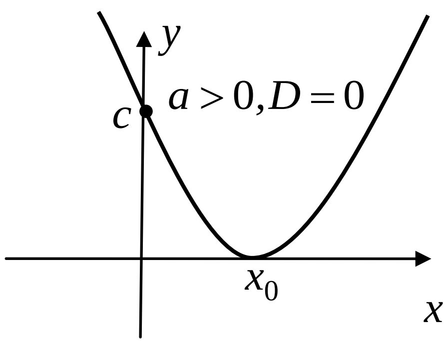


Fig. 3.24. **The parabola for** $a > 0$, $D = 0$

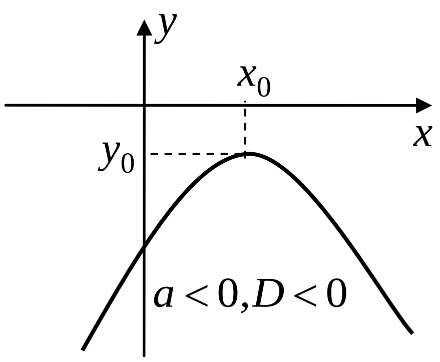


Fig. 3.25. **The parabola for** $a < 0$, $D < 0$

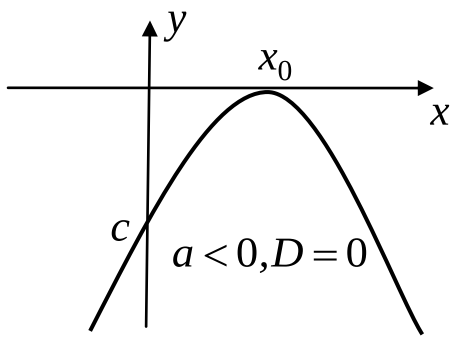


Fig. 3.26. **The parabola for** $a < 0$, $D = 0$

Example 3.6. Build the function graphs:

a) $y=\frac{1}{2}x^2$; b) $y=-x^2$; c) $y=x^2-6x+8$; d) $y=-x^2+4x+3$.

Solution: a) $y=\frac{1}{2}x^2$.

The graph of this function is a parabola, whose branches are directed upwards if the coefficient at x^2 is greater than zero. The function is even, so the graph is symmetric to the *y*-axis. For a more accurate construction, take a few points: (0; 0), (1; 1/2), (2; 2), (3; 9/2).

The resulting points are built in the coordinate system and connected by a smooth line (Fig. 3.27):

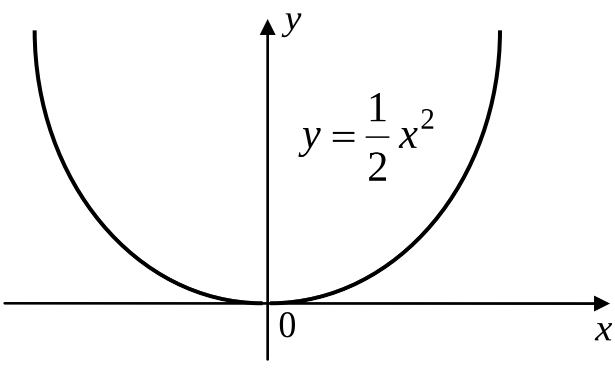


Fig. 3.27. **The parabola of the function** $y=\frac{1}{2}x^2$

b) $y=-x^2$.

Because the coefficient at x^2 is negative, the parabola branches are directed downwards. The function is even, so the graph is symmetric to the

y-axis. Take several points lying on this parabola: (0; 0), (1; −1), (2; −4), (3; −9). The resulting points are connected by a smooth line. (Fig. 3.28).

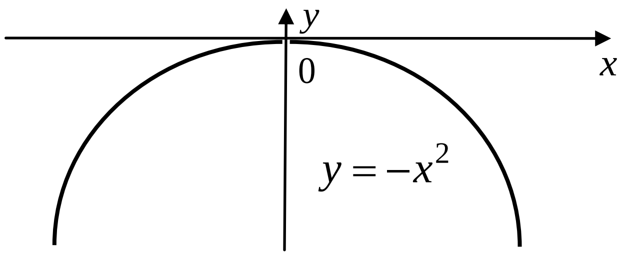


Fig. 3.28. **The parabola of the function** $y = -x^2$

c) $y = x^2 - 6x + 8$. To build the graph of this function, select the full square in the square trinomial $x^2 - 6x + 8$:

$$x^2 - 6x + 8 = x^2 - 2 \cdot 3 \cdot x + 9 - 9 + 8 = (x-3)^2 - 1.$$

Thus, the equation takes the form: $y = (x-3)^2 - 1$, whence it follows that the coordinates of the vertex of the parabola are: $x_0 = 3, y_0 = -1.$

Find the zeros of the function $y = 0$:

$$(x-3)^2 - 1 = 0, \text{ that is } (x-3)^2 = 1, \text{ or } |x-3| = \pm 1, \text{ then } x_1 = 2, x_2 = 4.$$

For $x = 0$ the parabola intersects the y-axis at $y = 8$.

Now, we plot the function graph (Fig. 3.29).

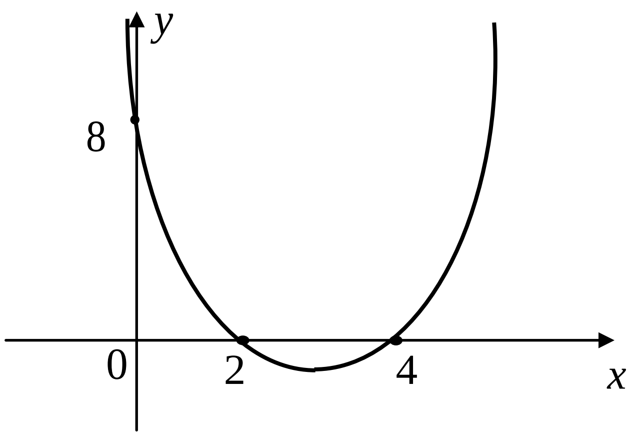


Fig. 3.29. **The parabola of the function** $y = x^2 - 6x + 8$

d) $y=-x^2+4x+3$. As in the previous example, select the full square in the trinomial $-x^2+4x+3$:

$$-x^2+4x+3=-\left(x^2-4x-3\right)=-\left(x^2-2\cdot 2x+4-4-3\right)=-\left((x-2)^2-7\right)=$$
$$=-(x-2)^2+7.$$

So, the function can be represented as follows:

$$y=-(x-2)^2+7,$$

whence it follows that the coordinates of the vertex of the parabola are $x_0=2, y_0=7$.

For $x=0$ the parabola intersects the y-axis at $y=3$.

Find the zeros of the function $y=0$:

$$-x^2+4x+3=0,\ x^2-4x-3=0.$$

Then the roots are: $x_{12}=2\pm\sqrt{4+3}$, $x_1=2-\sqrt{7}, x_2=2+\sqrt{7}$.

The branches of the parabola are directed downwards, because the coefficient at x^2 is less than zero.

Now we can plot the function graph (Fig. 3.30):

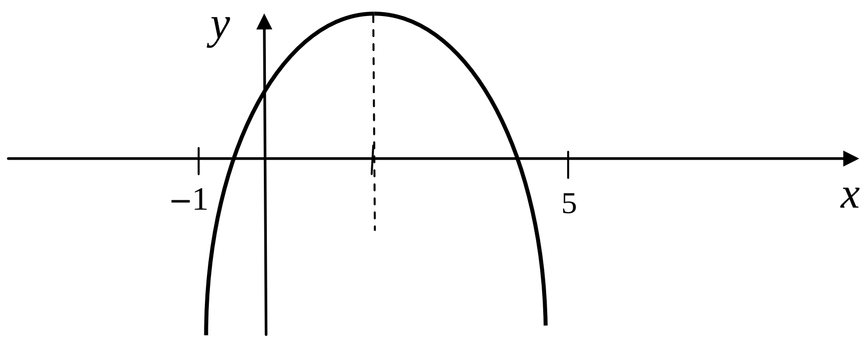


Fig. 3.30. **The parabola of the function** $y=-x^2+4x+3$

The function $y=x^n, n\in N$

The power function $y=x^n, n\in N$ is defined on the whole number line, that is $x\in R$.

If $x=0$, then $y=0$, therefore, the graph of this function passes through the origin.

With an even exponent $(n = 2m, m \in N)$, the power function is even (its graph is symmetric to the y-axis) and can take values from the interval y > 0. Its graphs are parabolas of the second, fourth, and so on orders (Fig. 3.31).

With an odd exponent $(n = 2m+1, m \in N)$, the function $y = x^n$ is odd (its graph is symmetric with respect to the coordinate origin) and takes the values $y \in [-\infty; +\infty)$. Its graphs are the parabolas of the third, fifth, etc. orders (Fig. 3.32).

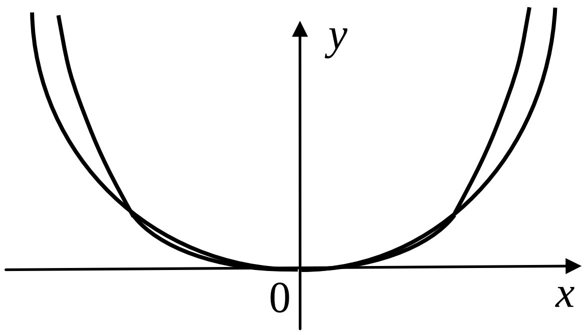


Fig. 3.31. **Even parabolas**

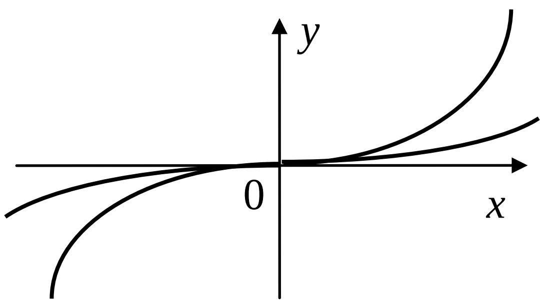


Fig. 3.32. **Odd parabolas**

The function $y = x^{-n}, n \in N$

The domain of this function is $D(y) = R \setminus \{0\}$.

For $n = 2m, m \in N$ the function $y = \frac{1}{x^{2m}}$ is even, so the graph is symmetrical to the axis Oy.

The function takes the values $E(y) = (0; +\infty)$, that is, the graph is above the Ox axis.

If $x < 0$, the function increases; if $x > 0$, the function decreases. The graph is depicted in Fig. 3.33:

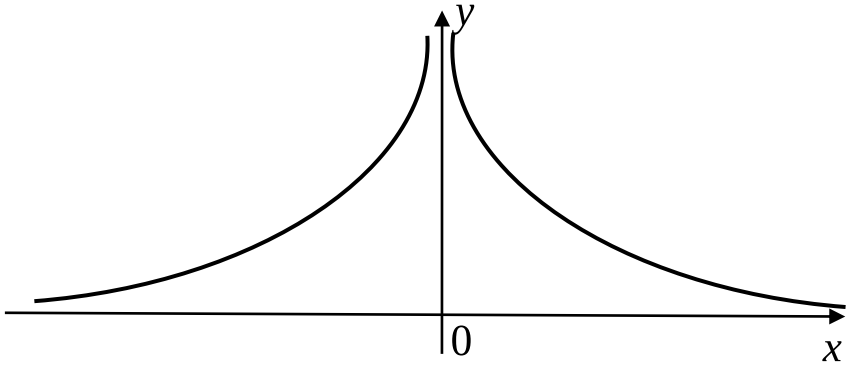


Fig. 3.33. **Even exponents**

For $n = 2m+1, m \in N$ the function $y = \frac{1}{x^{2m+1}}$ is odd, therefore, the graph is symmetric to the origin. The function takes the values $E(y) = (-\infty;0) \cup (0;+\infty)$.

If $x < 0$, the function decreases; if $x > 0$, the function increases. The graph is depicted in Fig. 3.34.

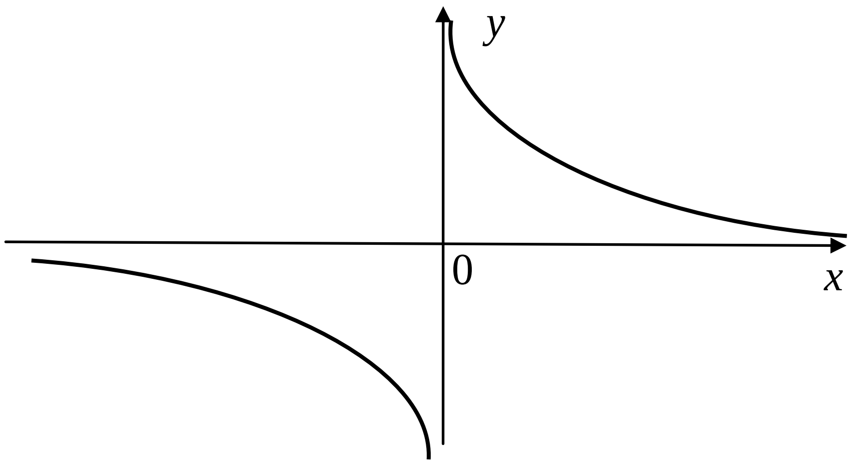


Fig. 3.34. **Odd exponents**

The function $y = \sqrt{x}$

The function exists on the set $D(y) = R \setminus \{0\}$ and takes the values $E(y) = (0;+\infty)$, that is, the graph is located in the first quarter. This function is a general form function. $y = \sqrt{x}$ is an increasing function. For a more accurate construction, take a few points: (0; 0), (1; 1), (4; 2), (9; 3).

The graph is depicted in Fig. 3.35:

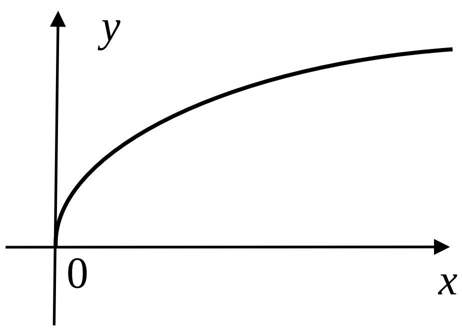


Fig. 3.35. **The function** $y = \sqrt{x}$ **graph**

The function $y=\sqrt[3]{x}$

The domain is $D(y)=(-\infty;+\infty)$. The codomain is $E(y)=(-\infty;+\infty)$. If $x=0$, then $y=0$, that is, the graph of the function passes through the origin. The function is odd, so the graph is symmetric to the origin. In the domain of definition, the function increases. For a more accurate construction, take a few points. The function graph is shown in Fig. 3.36.

To any admissible set of the variable x the function $y=f(x)$ gives a certain number y (Fig. 3.36).

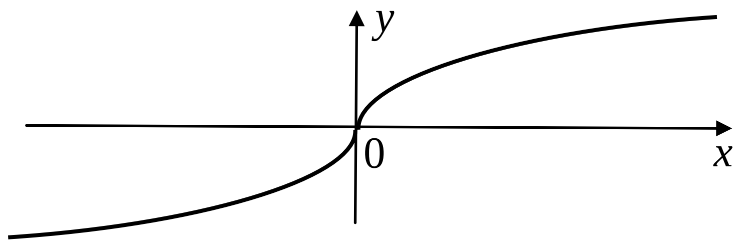


Fig. 3.36. **The function $y=\sqrt[3]{x}$ graph**

3.3. The inverse functions

Example 3.7. To each value y, the equality $y=2x+3$ assigns the following value x: $x=\dfrac{y-3}{2}$, which is the needed inverse function.

Example 3.8. To each value y, the equality $y=x^3$ assigns the following value x: $x=\sqrt[3]{y}$, which is the needed inverse function.

A function that is determined from the functional equation $y=f(x)$ if y is considered to be an independent variable, but x to be the function: $x=\varphi(y)$, is called inverse to the function $y=f(x)$. If the function φ is inverse to the function f, then the function f is inverse to the function φ.

Suppose we have a graph of the function $y=f(x)$. The same graph can serve as a graph of the inverse function, because both equations $y=f(x)$ and $x=\varphi(y)$ give the same functional relationship between x and y, but in the latter case, the values of the argument are considered on the Oy-axis, and the values of the function – on the Ox-axis.

In standard notations when x is denoted as an independent variable and y is denoted as a dependent variable, the function inverse to the function $y = f(x)$ can be written as $y = \varphi(x)$.

In this case, the graph of the function $y = \varphi(x)$ will be symmetrical to the graph of the function $y = f(x)$ with respect to the bisector of the first and third coordinate angles.

It should be noted that the domain of the function $y = f(x)$ coincides with the set of values of the inverse function, and vice versa (Fig. 3.37):

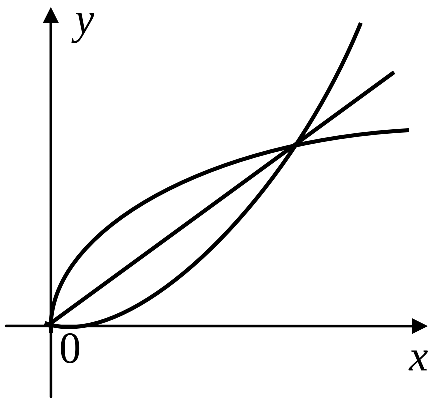


Fig. 3.37. **The inverse functions**

Example 3.9. Find the function inverse to the given function $y = 3x - 2$ and plot the graphs of both functions.

Solution: By solving this equation for x, we get the inverse function $x = \dfrac{y+2}{3}$.

Replacing in the latter x by y and y by x, we get the inverse function:

$$y = \frac{x+2}{3}.$$

Plot graphs of the functions $y = 3x - 2$ and $y = \dfrac{x+2}{3}$. Each graph of these functions represents a straight line, which is defined by two points.

For the straight line $y = 3x - 2$ we have: (0: −2), (2/3; 0).

For the straight line $y = \dfrac{x+2}{3}$ we have: (0; 2/3), (−2; 0).

The graphs of these functions are depicted in Fig. 3.38. They are symmetrical with respect to the bisector $y = x$.

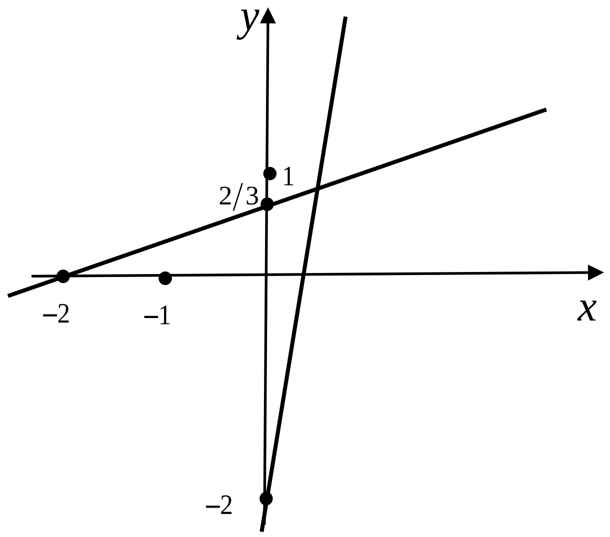


Fig. 3.38. **The direct and inverse functions**

3.4. The general ideas about non-elementary functions. Function graphs

1. $y = |x|$. Using the definition of a module, we have:

$$y = \left[\begin{array}{ll} x, & x \geq 0, \\ -x, & x < 0. \end{array}\right.$$

The graph of this function is depicted in Fig. 3.39.

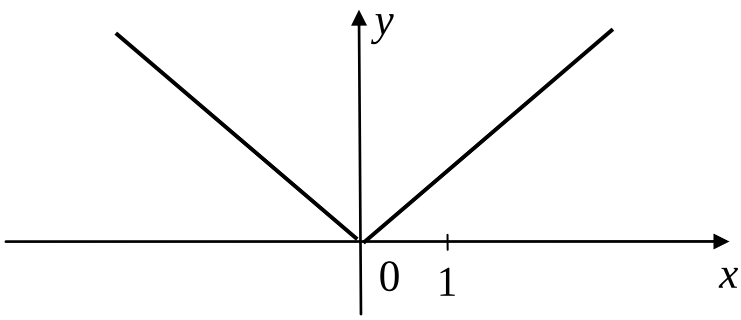


Fig. 3.39. **The generalized function** $y = |x|$

2. $y=[x]$ – the whole number of x is defined as the largest integer not exceeding x.

For example, $[5{,}18]=5$, $[0{,}6]=0$, $[-15{,}3]=-16$.

The graph of $y=[x]$ is depicted in Fig. 3.40.

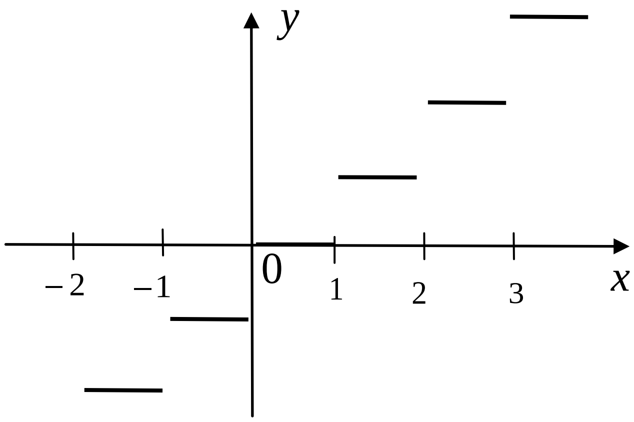


Fig. 3.40. **The generalized function** $y=[x]$

3. $y=\{x\}$ – the fractional part of x: $\{x\}=x-[x]$.

For example, $\{2{,}4\}=2{,}4-2=0{,}4$, $\{-3{,}7\}=-3{,}7-(-4)=0{,}3$.

The graph of $y=\{x\}$ is depicted in Fig. 3.41.

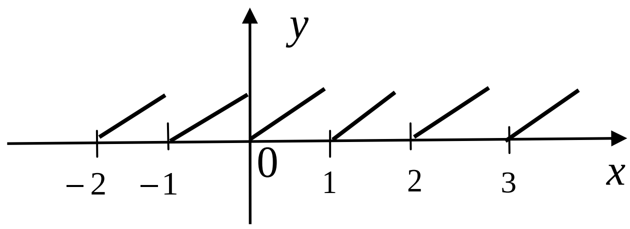


Fig. 3.41. **The generalized function** $y=\{x\}$

3.5. Plotting the functions using geometric transformations of the known graphs

The basic rules for constructing a graph of a function using geometric transformations

1. The graph of the function $y=f(x)+a$ is obtained from the graph of the function $y=f(x)$ with the aid of the parallel transfer in the positive

direction of the y-axis by a for $a>0$ and in the negative direction of the axis Oy by $|a|$ for $a<0$.

2. The graph of the function $y=f(x+a)$ is obtained from the graph of the function $y=f(x)$ with the aid of the parallel transfer in the negative direction of the axis Ox by a for $a>0$ and in the positive direction of the axis Ox by $|a|$ for $a<0$.

3. The graph of the function $y=-f(x)$ is obtained from the graph of the function $y=f(x)$ by symmetric display about the axis Ox.

4. The graph of the function $y=f(-x)$ is obtained from the graph of the function $y=f(x)$ by symmetric display about the axis Oy.

5. The graph of the function $y=|f(x)|$ is obtained from the graph of the function $y=f(x)$ as follows: the part of the graph of the function that lies above the axis Ox remains unchanged, while the part of it that lies below the axis Ox is displayed symmetrically about the axis Ox, that is:

$$|f(x)|=\left[\begin{array}{ll} f(x), & f(x)\ge 0, \\ -f(x), & f(x)<0. \end{array}\right.$$

6. The graph of the function $y=f(|x|)$ is obtained from the graph of the function $y=f(x)$ by the following manipulations: for $x\ge 0$ the function graph is saved, for $x<0$ the function graph is displayed symmetrically about the axis Oy, that is:

$$f(|x|)=\left[\begin{array}{ll} f(x), & x\ge 0, \\ f(-x), & x<0. \end{array}\right.$$

7. The graph of the function $y=f(kx)$, where $k>0$, is obtained from the graph of the function $y=f(x)$ by k times compression to the axis Oy for $k>1$ and $1/k$ times stretching from the axis Oy for $k<1$.

8. The graph of the function $y=kf(x)$, where $k>0$, is obtained from the graph of the function $y=f(x)$ by k times stretching from the axis Ox for $k>1$ and $1/k$ times compression to the axis Ox for $k<1$.

Using geometric transformations of basic elementary functions, construct graphs of the following functions.

Example 3.10. 1) $y=(x-2)^2$; 2) $y=x^2+1$; 3) $y=-2x^3$; 4) $y=-x^2+4x+1$.

Solution: 1) $y=(x-2)^2$. Using the graph of the function $y=x^2$ and rule (2) we build the graph of this function by parallel transfer to two units of the graph of the function $y=x^2$ to the right (Fig. 3.42).

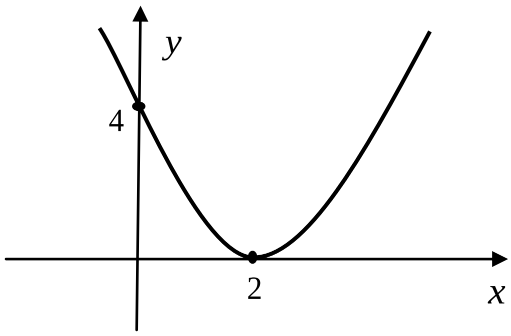


Fig. 3.42. **The function** $y=(x-2)^2$ **graph**

2) $y=x^2+1$. Using the graph of the function $y=x^2$ and rule (2) we build the graph of this function by parallel transfer to one unit of the graph of the function $y=x^2$ up along the axis Oy (Fig. 3.43).

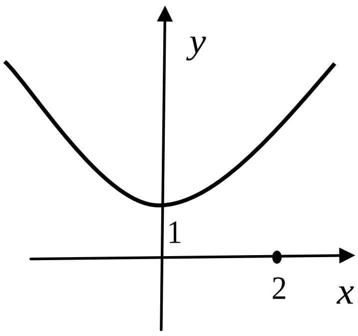


Fig. 3.43. **The function** $y=x^2+1$ **graph**

3) $y=-2x^3$. Using the graph of the function $y=x^3$, and rules (3) and (8), we build the graph of this function (Fig. 3.44).

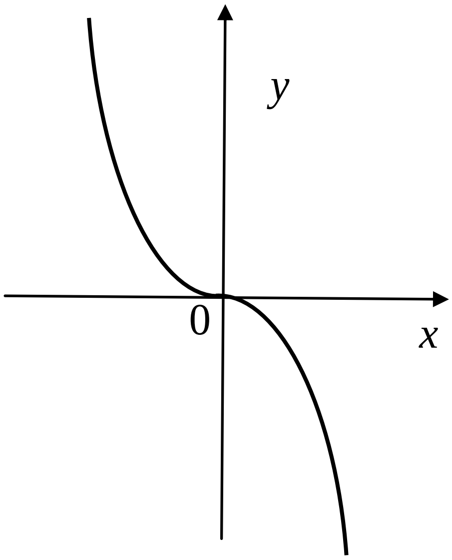


Fig. 3.44. **The function** $y = -2x^3$ **graph**

4) $y = -x^2 + 4x + 1$. First we select the full square in the right hand side:

$$-x^2 + 4x + 1 = -(x^2 - 4x - 1) = -((x-2)^2 - 5) = -(x-2)^2 + 5.$$

Then the function takes the form:

$$y = -(x-2)^2 + 5.$$

To build the graph of this function, we use the graph of the function $y = x^2$ and also rules (3) and (1) (Fig. 3.45).

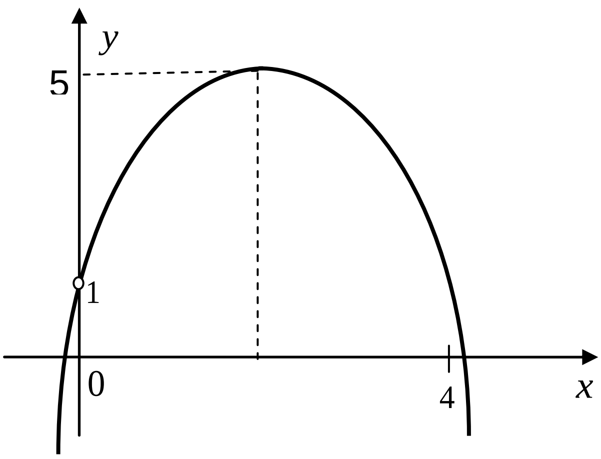


Fig. 3.45. **The function** $y = -x^2 + 4x + 1$ **graph**

Example 3.11. Build the following graphs:

1) $y = |x| - 3$; 2) $y = |x+2|$; 3) $y = x^2 - 4|x|$; 4) $y = |1 - 2x - x^2|$.

Solution: 1. $y = |x| - 3$. Using rules (5) and (1), we can build the graph of this function (Fig. 3.46).

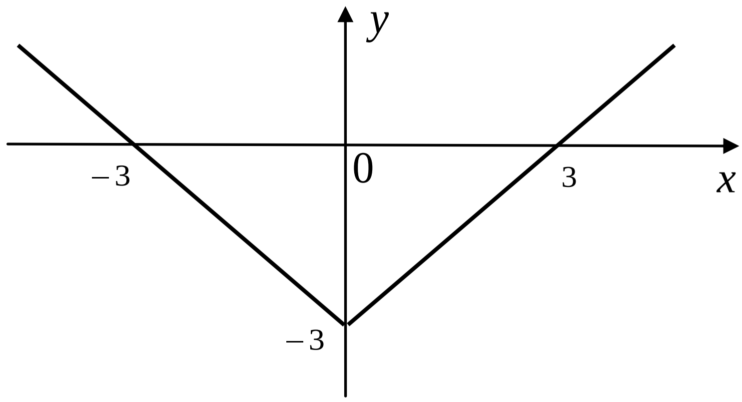


Fig. 3.46. **The function** $y = |x| - 3$ **graph**

2. $y = |x+2|$. Using rules (2) and (5) we can build the graph of this function (Fig. 3.47).

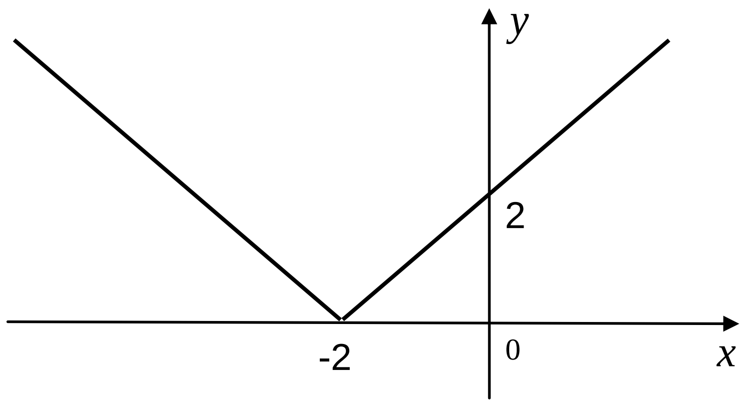


Fig. 3.47. **The function** $y = |x+2|$ **graph**

3. $y = x^2 - 4|x|$. To construct the graph of this function, we consider two cases, given that

$$|x| = \left[\begin{array}{ll} x, & x \geq 0, \\ -x, & x < 0. \end{array}\right.$$

For $x \geq 0$ we have: $y = x^2 - 4x = (x-2)^2 - 4$.

For $x < 0$ we have: $y = x^2 + 4x = (x+2)^2 - 4$.

Using rules (2) and (1) we can build the graph of this function $y = x^2 - 4|x|$ (Fig. 3.48).

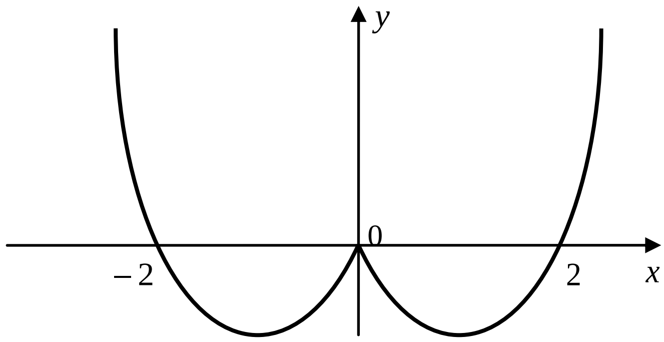


Fig. 3.48. **The function** $y = x^2 - 4|x|$ **graph**

4. $y = |1 - 2x - x^2|$. Let us transform the expression under the sign of the module: $1 - 2x - x^2 = -(x^2 + 2x - 1) = -((x+1)^2 - 2) = -(x+1)^2 + 2$; then $y = |-(x+1)^2 + 2|$. Using rules (2), (3), (1) and (5) we can build the graph of this function (Fig. 3.49):

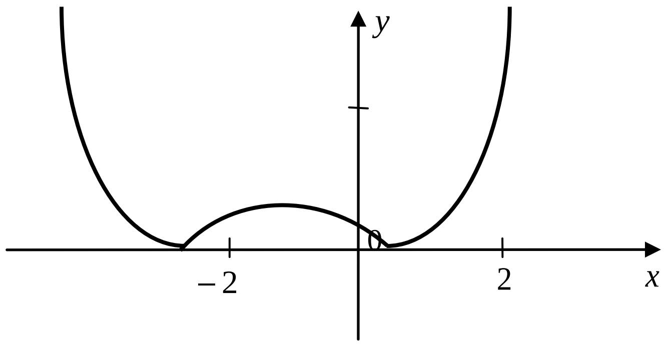


Fig. 3.49. **The function** $y = |1 - 2x - x^2|$ **graph**

Tasks for individual work

3.12. Find the domains: 1) $y = \dfrac{5x-1}{x-2}$; 2) $y = \dfrac{x+6}{x^2+x+1}$;

3) $y=\sqrt{5-10x}$; 4) $y=2\sqrt{x-1}-\frac{5}{\sqrt{4-x}}$; 5) $y=\sqrt{x^2+2x+4}$;

6) $y=log_2\left(-3x^2-7x-2\right)$; 7) $y=\sqrt{2+\frac{x}{1-x}}$; 8) $y=lg\,\frac{x(x-3)}{2x-5}$;

9) $y=2^{-\sqrt{-x}}$; 10) $y=3^{\sqrt{x}}+3^{\sqrt{-x}}$.

3.13. Find out which functions are even, odd or of general form:

1) $f(x)=x^2-2x^4+1$; 2) $f(x)=\frac{cos\,x}{x}$;. 3) $f(x)=x\cdot sin\,x$;

4) $f(x)=3^x+3^{-x}$; 5) $f(x)=x+sin\,x$; 6) $f(x)=tg2x+ctg2x$;

7) $f(x)=\sqrt[3]{x-1}$; 8) $f(x)=lg\left(x^2+1\right)$; 9) $y=x^2-2x$.

3.14. Build the graphs:

1) $y=2x+3$; 2) $y=-2x+6$; 3) $y=3x-6$;

4) $y=-3x-6$; 5) $y=5$; 6) $y=-2$.

3.15. Build the graphs:

1) $y=\frac{3}{x}$; 2) $y=-\frac{2}{x}$; 3) $y=\frac{4}{x}$; 4) $y=-\frac{5}{x}$.

3.16. Build the graphs:

1) $y=-4x^2$; 2) $y=x^2+8x-9$; 3) $y=x^2-4x+3$;

4) $y=-x^2+6x-5$; 5) $y=4x^2+16x-12$; 6) $y=-x^2+2x-1$.

3.17. Find the inverse functions:

1) $y=10x+10$; 2) $y=5x-2$; 3) $y=\sqrt{x}$; 4) $y=x^3$.

Build the graphs of the inverse functions and of the functions themselves:

3.18. $y=(x+3)^2$. **3.19.** $y=x^2+1$. **3.20.** $y=x^2-2x-4$.

3.21. $y=2x^2+4x-1$. **3.22.** $y=(x-1)^3$. **3.23.** $y=x^3-1$.

3.24. $y=\sqrt{x-2}$. **3.25.** $y=\sqrt{x+2}$. **3.26.** $y=\sqrt{2-x}$.

3.27. $y=\sqrt{x}+1$. **3.28.** $y=\sqrt{x}-2$. **3.29.** $y=-\sqrt{x}+4$.

3.30. $y=\frac{1}{x-2}$. **3.31.** $y=\frac{1}{x+1}$. **3.32.** $y=\frac{1}{x}+1$. **3.33.** $y=2-\frac{1}{x}$.

3.34. $y=\frac{x-1}{x+1}$. **3.35.** $y=\frac{2-x}{2+x}$. **3.36.** $y=2^{x+1}$. **3.37.** $y=\left(\frac{1}{2}\right)^{x-1}$.

3.38. $y=|x|+1$. **3.39.** $y=-|x|+2$. **3.40.** $y=x^2-2|x|$.

3.41. $y=4|x|-x^2-3$. **3.42.** $y=\dfrac{x}{|x|}$. **3.43.** $y=\sqrt{|x|}$. **3.44.** $y=\sqrt{|x|}-2$.

Questions for self-assessment

1. What is the concept of a function?
2. What is the domain of a function?
3. What is the codomain of a function?
4. Elementary functions and their graphs.
5. Inverse functions, their properties.
6. A generalized function, its properties.

4. Algebraic equations

4.1. Linear equations and equations reducible to them

The ability to solve linear equations is among those that each high school graduate should possess. However, solutions to them may present various kinds of difficulties.

Indeed, the simplest linear equation $ax+b=0$ 1) can have a unique solution for $a\neq 0$, namely: $x=-b/a$; 2) may have no solution for $a=0$ and $b\neq 0$, namely: $x\in\varnothing$; 3) may have an infinite set of solutions for $a=0$ and $b=0$, namely: $x\in R$.

Example 4.1. Investigate the solution to the equation

$$a(ax-2)=9x-6.$$

Solution: Transform the equation $x\cdot\left(a^2-9\right)=2(a-3)$.

If $a^2-9\neq 0$, that is $a\neq\pm3$, the equation has a unique solution:

$$x=\frac{2(a-3)}{a^2-9}=\frac{2}{a+3}.$$

If $a=3$, we have: $x\cdot(9-9)=2(3-3)$ or $x\cdot 0=0$, which means that the equation has an infinite number of solutions: $x\in R$.
If $a=-3$, we have the equation $x\cdot(9-9)=2(-3-3)$, that is: $x\cdot 0=-12$. This equation has no solution.

Answer: $x=\dfrac{2}{a+3}$, if $a\neq\pm 3$; $x\in R$, if $a=3$, $x\in\varnothing$, if $a=-3$.

Example 4.2. Solve the equation:

$$\frac{x-4}{x+4}+\frac{16x}{x^2-16}=\frac{x+4}{x-4}.$$

Solution: This is a fractional rational equation containing an unknown in the denominator. The domain is $x\neq\pm 4$. After completing the transformations, we get:

$$\begin{cases}\dfrac{(x-4)^2-(x+4)^2+16x}{x^2-16}=0,\\ x-4\neq 0,\\ x+4\neq 0,\end{cases}\Leftrightarrow\begin{cases}\dfrac{0\cdot x}{x^2-16}=0,\\ x\neq 4, x=-4.\end{cases}\Leftrightarrow\begin{cases}0\cdot x=0,\\ x\neq\pm 4.\end{cases}$$

The solution is any real number, except for $x=4$ and $x=-4$.

Answer: $x\in R\setminus\{-4;4\}$.

Example 4.3. Solve the equation:

$$\frac{1}{x-7}-\frac{1}{x-5}-\frac{1}{x-6}+\frac{1}{x-4}=0.$$

Solution: The domain is $x\neq 4;5;6;7$. Combine the fractions and perform the actions inside them:

$$\left(\frac{1}{x-7}+\frac{1}{x-4}\right)=\left(\frac{1}{x-5}+\frac{1}{x-6}\right).$$

We obtain: $$\frac{2x-11}{(x-7)\cdot(x-4)}=\frac{2x-11}{(x-5)\cdot(x-6)}.$$

The equality in the numerators gives the first equation:

1) $2x-11=0,\ x=11/2=5{,}5$.
2) After reducing to $2x-11$ we get the second equation:

$$(x-7)\cdot(x-4)=(x-5)\cdot(x-6),\ x^2-11x+28=x^2-11x+30,$$

from which it follows that the second equation is reduced to the equation $28=30$ that has no solutions.

Answer: $\{5{,}5\}$.

Example 4.4. Investigate the equation:

$$\frac{ax}{a^2-1}+\frac{3x+2}{a+1}=\frac{1}{a-1}.$$

Solution: This equation has no meaning for $a=\pm1$. Let us investigate it for the remaining values of the parameter a. After transformations, the equation takes the form:

$$(4a-3)x=3-a.$$

The following cases are possible:

1. If $4a-3\neq0$, that is $a\neq\frac{3}{4}$, the solution will be: $x=\frac{3-a}{4a-3}$.

2. If $4a-3=0$, that is $a=3/4$, we have a false equality:

$$0\cdot x=3-\frac{3}{4},$$

wherefrom it follows that there are no solutions: $x\in\varnothing$.

Answer: For $a=\pm1$ the equation has no sense; for $a\neq\pm1$ and $a\neq3/4$, the solution is: $x=\frac{3-a}{4a-3}$; for $a=3/4$ there are no solutions.

Example 4.5. Investigate the equation:

$$\frac{x}{a+x}+\frac{x+a}{a-x}=\frac{x+a-2a^2}{a^2-x^2}.$$

Solution: The domain is $x \neq \pm a$. After the transformation of the left side, the equation is reduced to:

$$\frac{3ax + a^2}{a^2 - x^2} = \frac{x + a - 2a^2}{a^2 - x^2}.$$

Equal numerators imply:

$$\begin{cases} 3ax + a^2 = x + a - 2a^2 \\ x \neq \pm a \end{cases}.$$

From the first equation we have: $x(3a - 1) = a(1 - 3a)$. Let us study this one:

a) if $3a - 1 \neq 0$, that is $a \neq 1/3$, then $x = -a$, which is contrary to the requirement $x \neq -a$. This means that this equation has no solution for $a \neq 1/3$;

b) if $3a - 1 = 0$, that is $a = 1/3$, we have the equality: $x \cdot 0 = (1/3) \cdot 0$, which is fulfilled for $x \in R$, subject to the requirements $x \neq \pm a$. That is: $x \in R \setminus \{\pm 1/3\}$.

Answer: for $a = 1/3$, $x \in R$, besides $x \neq \pm 1/3$; for $a \neq 1/3$, there are no solutions.

Example 4.6. Solve the equation: $||7 - 2x| - 3| = 5$.

Solution: According to the definition of the module, this equation splits into two equations.

1. $|7 - 2x| - 3 = 5$;
2. $|7 - 2x| - 3 = -5$.

Let us solve each of the equations and combine the results:

1. $|7 - 2x| = 8$, wherefrom: $7 - 2x = \pm 8$, and $x = \frac{7 \pm 8}{2}$, that is:

$$x_1 = \frac{15}{2},\ x_2 = -\frac{1}{2}.$$

2. $|7 - 2x| = -2$, which means, due to the property of the module, that there are no solutions.

Answer: $\left\{-\frac{1}{2};\ \frac{15}{2}\right\}$.

Example 4.7. Solve the equation: $\left|x-7\right|+x=8$.

Solution: The equation can be reduced to two systems:

$$1)\ \begin{cases} x-7\geq 0, \\ x-7+x=8, \end{cases} \Leftrightarrow \begin{cases} x\geq 7, \\ x=\dfrac{15}{2}, \end{cases} \Rightarrow x=7{,}5;$$

$$2)\begin{cases} x-7<0, \\ -x+7+x=8, \end{cases} \Leftrightarrow \begin{cases} x<7, \\ 7=8, \end{cases} \Rightarrow \text{there are no solutions.}$$

Answer: $\{7{,}5\}$.

Example 4.8. Solve the equation: $\left|3x+6\right|-\left|x-1\right|=x+2$.

Solution: Let us apply the interval method to solve the equation. On the numerical axis, we mark the points $x=1$ and $x=-2$ at which each of the expressions under the module vanishes. We form three intervals and solve the equation in each of the obtained intervals (Fig. 4.1).

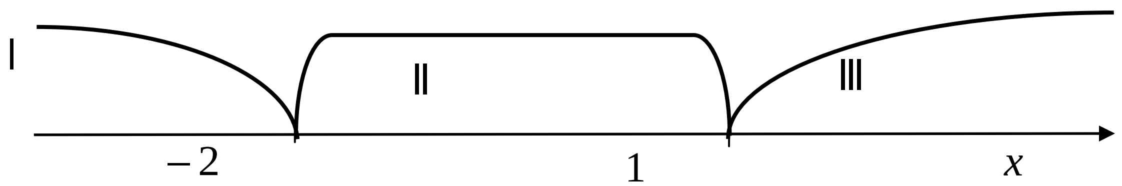


Fig. 4.1. **The method of intervals applied to the equation**
$\left|3x+6\right|-\left|x-1\right|=x+2$

I $\begin{cases} x<-2, \\ -3x-6+x-1=x+2; \end{cases}$ $\begin{cases} x<-2, \\ x=-3; \end{cases}$ $x=-3$;

II $\begin{cases} -2\leq x<1, \\ 3x+6+x-1=x+2; \end{cases}$ $\begin{cases} -2\leq x<1, \\ x=-1; \end{cases}$ $x=-1$;

III $\begin{cases} x\geq 1, \\ 3x+6-x+1=x+2; \end{cases}$ $\begin{cases} x\geq 1, \\ x=-5; \end{cases}$ $x\in\varnothing$.

Answer: $\{-3;-1\}$.

Example 4.9. Solve the equation:

$$\sqrt{x^2+2x+1}-\sqrt{x^2-4x+4}=3.$$

Solution: Transform this equation:

$$\sqrt{(x+1)^2}-\sqrt{(x-2)^2}=3; \quad |x+1|-|x-2|=3.$$

To solve the linear equation obtained, we apply the method of intervals (Fig. 4.2), which was described in detail in the previous example. Then,

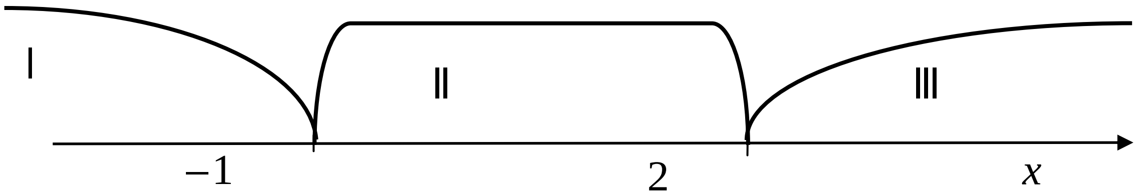


Fig. 4.2. **The method of intervals applied to the equation**

$$\sqrt{x^2+2x+1}-\sqrt{x^2-4x+4}=3$$

I $\begin{cases} x<-1, \\ -x-1+x-2=3; \end{cases}$ $\begin{cases} x<-1, \\ -3=3; \end{cases}$ $x\in\varnothing$;

II $\begin{cases} -1\le x<2, \\ x+1+x-2=3; \end{cases}$ $\begin{cases} -1\le x<2, \\ x=2; \end{cases}$ $x\in\varnothing$;

III $\begin{cases} x\ge 2, \\ x+1-x+2=3; \end{cases}$ $\begin{cases} x\ge 2, \\ 3=3; \end{cases}$ $x\ge 2$.

Combining the received results, we get: $x\ge 2$.

Answer: $x\in[2;\infty)$.

Tasks for individual work

Solve the equations:

4.10. $\dfrac{x-3}{8}+\dfrac{x+9}{12}=\dfrac{3x+7}{20}+3$.

4.11. $\dfrac{x-1}{4}-\dfrac{1}{8}\cdot\left(\dfrac{x-5}{4}-\dfrac{14-2x}{2}\right)=\dfrac{x-9}{2}-\dfrac{7}{8}$.

4.12. $\frac{x+1}{x-1}-\frac{x+2}{x+3}+\frac{4}{x^2+2x-3}=0$.

4.13. $\frac{x+1}{x-3}-\frac{x-2}{x+3}=\frac{3\cdot(3x-1)}{x^2-9}$.

4.14. $(2a-1)\cdot x=3a(a+2)x$.

4.15. $a^2x+2=a(x+2)$.

4.16. $m(mx-1)=3\cdot(mx-1)$.

4.17. $\frac{2}{n}-\frac{x}{n^2+3n}=\frac{1-2x}{n+3}$.

4.18. $0,3|x|-1=3-0,5|x|$.

4.19. $|x-2|=3$.

4.20. $||5-x|-4|=9$.

4.21. $|x|=x+2$.

4.22. $|-x+2|=2x+1$.

4.23. $|x-1|+|x+1|=2$.

4.2. Quadratic equations and equations reducible to them

The roots of the quadratic equation $ax^2+bx+c=0$ ($a\neq 0$) can be found by the formula:

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

where $b^2-4ac=D$ is the discriminant of the quadratic equation. If $D>0$, then the equation has two different roots; if $D=0$, the equation has two equal roots; if $D<0$, the equation has two real roots.

The roots of a quadratic equation x_1 and x_2, and also its coefficients are related by the Vieta's formula:

$$\begin{cases} x_1+x_2=-b/a, \\ x_1\cdot x_2=c/a. \end{cases}$$

For the equation $x^2+px+q=0$ the Vieta's formula is as follows:

$$\begin{cases} x_1+x_2=-p, \\ x_1\cdot x_2=q. \end{cases}$$

Example 4.24. Solve the equation:

$$\frac{x^2 - 3,5x + 1,5}{x^2 - x - 6} = 0.$$

Solution: This equation is equivalent to the following system:

$$\begin{cases} x^2 - 3,5x + 1,5 = 0, \\ x^2 - x - 6 \neq 0, \end{cases}$$

wherefrom after solving the quadratic equations we get only one root:

$$\begin{cases} \left[\begin{array}{l} x = 3, \\ x = 0,5; \end{array}\right. \\ \left[\begin{array}{l} x \neq 3, \\ x \neq -2; \end{array}\right. \end{cases} \Rightarrow x = 0,5.$$

Answer: $\{0,5\}$.

Example 4.25. Solve the equation:

$$\left|x^2 - 3x + 2\right| = 3x - x^2 - 2.$$

Solution: Let us rewrite the equation to the form:

$$\left|x^2 - 3x + 2\right| = -\left(x^2 - 3x + 2\right).$$

According to the module definition, we have:

$$x^2 - 3x + 2 \leq 0,$$

where the solution is $x \in [1;2]$.

Answer: $[1;2]$.

Example 4.26. Solve the equation: $x^2+3x+4|x-2|-8=0$.

Solution: Consider two possible cases in solving this equation.

1. Let $x-2\geq 0$, that is $x\geq 2$.

Then $|x-2|=x-2$ and this equation takes the form:

$$x^2+3x+4(x-2)-8=0 \Leftrightarrow x^2+7x-16=0,$$

wherefrom:

$$x_1=\frac{-7-\sqrt{113}}{2},\ x_2=\frac{-7+\sqrt{113}}{2}.$$

The roots obtained do not satisfy the condition $x\geq 2$. Because of that, these roots are redundant.

2. Let $x-2<0$, that is $x<2$. Then: $|x-2|=2-x$.

This equation, along with the requirement $x<2$, takes the form:

$$\begin{cases} x^2+3x+4(2-x)-8=0, \\ x<2, \end{cases} \Leftrightarrow \begin{cases} x^2-x=0, \\ x<2, \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x=0, \\ x=1, \end{array}\right. \\ x<2, \end{cases} \Leftrightarrow \{0;1\}.$$

Answer: $x=\{0;1\}$.

Example 4.27. For which values of a is the sum of the roots of the equation

$$x^2+\left(2-a-a^2\right)x-a=0$$

equal to zero?

Solution: By condition: $x_1+x_2=0$. Due to the Vieta's formula: $x_1+x_2=a^2+a-2$. Then we have the equation: $a^2+a-2=0$, from which we obtain: $a_1=1$, $a_2=-2$. It is necessary to check the discriminant sign of this equation for these values.

Because $2-a-a^2=0$, $D=\left(2-a-a^2\right)^2+4a=4a$. For $a=1$ $D=4>0$ but for $a=-2$, $D=-8<0$ (no solutions).

Answer: $a=\{1\}$.

Example 4.28. Find the value m, for which one of the roots of the equation $x^2+(2m-1)x+m^2+2=0$ is twice as large as the other.

Solution: By condition: $x_1=2x_2$. By the Vieta's formula, we have:

$$x_1+x_2=1-2m,\quad x_1\cdot x_2=m^2+2.$$

From these three equalities we find m. From the first two we have $3x_2=1-2m$, from which:

$$x_2=\frac{1-2m}{3},\quad x_1=\frac{2(1-2m)}{3}.$$

From the third equality we have:

$$\frac{2(1-2m)^2}{9}=m^2+2,$$

that is: $m^2+8m+16=0$; from which: $m=-4$.

Let us check the discriminant sign for $m=-4$:

$$D=(2m-1)^2-4(m^2+2)=-4m-7.$$

For $m=-4$, $D=16-7=9>0$.

Answer: $\{-4\}$.

Example 4.29. For which value of a is the difference of the equation

$$(a-2)\cdot x^2-(a-4)\cdot x-2=0$$

roots equal to 3?

Solution: By condition, $x_1-x_2=3$. By the Vieta's formula:

$$x_1+x_2=\frac{a-4}{a-2},\quad x_1\cdot x_2=-\frac{2}{a-2}.$$

Combining these equations we have:

$$x_1 = \frac{2a-5}{a-2},\ x_2 = \frac{1-a}{a-2}.$$

By substituting x_1 and x_2 in the last equality, we obtain:

$$\frac{2a-5}{a-2} \cdot \frac{1-a}{a-2} = -\frac{2}{a-2},$$

which can be transformed into $2a^2 - 9a + 9 = 0$. From this: $a_1 = 3$, $a_2 = \frac{3}{2}$.

Let us verify the discriminant signs for the values a_1 and a_2:

$$D = (a-4)^2 + 8 \cdot (a-2) = a^2 > 0.$$

Then, the answer is: $a = \{3/2; 3\}$.

Example 4.30. At what values of a do the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have a common root?

Solution: Let x_0 be a common root of these two equations. Then, obviously, the following equalities have to be fulfilled:

$$x_0^2 + ax_0 + 1 = 0,\ x_0^2 + x_0 + a = 0.$$

Subtracting, we obtain $ax_0 - x_0 + 1 - a = 0$, from which: $(a-1) \cdot x_0 = a - 1$. If $a \neq 1$, then $x_0 = 1$. Substituting $x_0 = 1$ in any equation, we obtain: $a = -2$.

If $a = 1$, both equation become the same: $x^2 + x + 1 = 0$. This one does not have any solution because of $D < 0$.

Answer: $\{-2\}$.

Tasks for individual work

Solve the equations:

4.31. $x^2 + 2103 \cdot x + 2102 = 0$. **4.32.** $\frac{x^2 + 3 \cdot x - 70}{x^2 - x - 6} = 0$.

4.33. $x^2 + 2{,}5 \cdot |x| - 1{,}5 = 0$. **4.34.** $\frac{x \cdot (2 \cdot x - 3)}{2} + \frac{(3 \cdot x - 1)^2}{5} - \frac{(x+3)^2}{5} = 1$.

4.35. $|x-3| = (x-3)^2$. **4.36.** $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{64}{x^2-16}$.

4.37. $\frac{2}{x^2 - x + 1} - \frac{1}{x+1} = \frac{2 \cdot x - 1}{x^3 + 1}$. **4.38.** $|x^2 - 4| = x^2 - 4$.

4.39. $|5 \cdot x - x^2 - 6| = x^2 - 5 \cdot x + 6$. **4.40.** $\frac{3|x| + 2}{|x| - 1} = 3$. **4.41.** $\left|\frac{3x+2}{x-1}\right| = 3$.

4.42. $\left|\frac{3|x| + 2}{|x| - 1}\right| = 3$. **4.43.** $x|3x+5| = 3x^2 + 4x + 3$.

4.44. $(1+x)|x+2| + x|x-3| = 6x + 2$.

4.45. For which values of m will the roots of the equation $(m-1)x^2 - 2(m+1)x + m + 4 = 0$ be equal?

4.46. Find m in the equation $2x^2 - 11x + m = 0$ if $2x_1 - x_2 = 2$, where x_1 and x_2 are the roots of the equation.

4.47. For which values of a will one of the roots of the equation $x^2 - \frac{15}{4}x + a^3 = 0$ be the square of another root?

4.48. For which values of a does the curve $y = (2a-5)x^2 - 2(a-1)x + 3$ touch the axis Ox?

4.49. For which values of a is the sum of the roots of the equation $x^2 - 2a(x-1) - 1 = 0$ equal to the sum of the squares of its roots?

4.50. For which values of k is one root of the equation $(k^2 - 5k + 3)x^2 + (3k-1)x + 2 = 0$ a half of the other one?

4.3. Problems on compilation of linear and quadratic equations

Consider the examples of solving problems using linear and quadratic equations.

Example 4.51. Eight workers completed work in 6 days. For how many days would 12 workers do the same work with the same labor productivity?

Solution: This task concerns the proportional division. Obviously, the number of workers and the time for doing the work are inversely proportional. Let 12 workers do a given job in x days. Have a proportion:

$$\frac{8}{12} = \frac{x}{6}, \text{ wherefrom } x = \frac{8 \cdot 6}{12} = 4.$$

Answer: 4 days.

Example 4.52. A two-digit number was conceived, for which the first digit is 2 less than the second one. If this number is divided by the sum of its digits, then the quotient will turn out 4 and 6 in the residual. What number was conceived?

Solution: Let the second digit of the conceived number be x, then, by condition, the first digit is $(x-2)$. Then, the number will be $10(x-2)+x$ or $11x-20.$

The sum of the digits is $(x-2)+x = 2x-2$. After dividing $11x-20$ by $2x-2$ the quotient will be 4 with a remainder of 6.

From that we have the equation: $11x-20 = 4(2x-2)+6$, or $3x = 18$, wherefrom $x = 6$ and the first digit will be $x-2 = 4$. That is, the conceived number was 46.

Answer: 46.

Example 4.53. Sea water contains 5 % (by weight) of salt. How many kg of fresh water do you need to add to 40 kg of sea water for the salt content in the mixture to be 2 %?

Solution: Suppose that you need to add x kg of fresh water to 40 kg of sea water. Then the weight of the mixture will be $(40+x)$ kg, in which, by condition, there must be 2 % of salt, that is, $(40+x)\cdot 0{,}02$ kg. In sea water there was $40 \cdot 0{,}05$ kg of salt. Since the amount of salt in the sea water and the mixtures is the same, the following equation can be made up: $(40+x)\cdot 0{,}02 = 40 \cdot 0{,}05$,

wherefrom: $x = 60$.

Answer: 60 kg.

4.54. The pool is filled with two pipes in 6 hours. One pipe can fill a pool 5 hours faster than another. For how many hours can the pool be filled by each pipe separately?

Solution: Let the first pipe fill the pool in x hours, the other – in $(x-5)$ hours, that is $x>5$. Let the pool volume be 1. Then, in an hour, the first pipe fills the $\frac{1}{x}$ part of the pool, the other – the $\frac{1}{x-5}$ part. The two pipes together fill the $\left(\frac{1}{x}+\frac{1}{x-5}\right)$ part of the pool in an hour. By condition, together they fill the pool in 6 hours. Then we have the equation:

$$\left(\frac{1}{x}+\frac{1}{x-5}\right)\cdot 6=1,$$

which gives us the square equation:

$$x^2-17x+30=0.$$

Solving this equation, we find for the first pipe separately: $x=15$ or $x=2$. Because $x>5$, $x=2$ is not suitable for the purpose of the task. For the second pipe separately we have $15-5=10$ hours.

Answer: 15 hours and 10 hours.

Example 4.55. The truck was delayed 12 minutes at a gas station. Then, having increased the speed by 15 km/h, at a distance of 60 km, it made up for the lost time. Find the initial vehicle speed.

Let the initial vehicle speed be x km/h. Then, the 60 km distance would be covered in $\frac{60}{x}$ hours. But its speed was $(x+15)$ km/h so it takes the truck only $\frac{60}{x+15}$ hours. The delay time was $\frac{1}{5}$ hours. This one gives us the equation:

$$\frac{60}{x}-\frac{60}{x+15}=\frac{1}{5}.$$

After the transformation, we have the quadratic equation: $x^2+15x-4500=0$.

Solution: $x_1 = 60$, $x_2 = -75$. According to the task, $x > 0$, so $x = 60$ km/h is the initial speed.

Answer: 60 km/h.

Tasks for individual work

4.56. For tailoring 6 tents you need 120 m of tarpaulin 1.2 m wide. How many meters of tarpaulin 1.5 m wide are needed to sew these tents?

4.57. 21 % cream is obtained from milk and 24 % butter by weight from cream. How much milk should we take to get 630 kg of butter?

4.58. What amount of 26 % sulfuric acid needs to be mixed with 40 kg of 68 % acid to produce acid of 32 % concentration?

4.59. In a two-digit number, the digit of units is 3 more than the digit of tens. If we add to this number a number in which the places of digits of ones and tens are swapped, we get 143. Find this number.

4.60. Find a two-digit number, knowing that the number of tens is twice the number of ones, and that the product of this number by the sum of its digits is 144.

4.61. The fee for the book was distributed between the two authors in the ratio of 8 : 6. If this fee were distributed in the ratio of 7 : 5, then one of the authors would receive 25 UAH more than he actually received. What is the amount of the fee?

4.62. Two workers can do a given job together in 12 days. In how many days will each worker perform the job separately, if the first worker can do this work 10 days faster than the second one?

4.63. The first worker can do some work 4 hours faster than the second one. Initially, they worked together for 2 hours, after which the rest of the work was done by the first worker on their own in 1 hour. How many hours can the second worker do all the work?

4.64. The product of two consecutive positive integers is 552. What are these numbers?

4.65. The numerator of a fraction is 2 less than its denominator. If you add up this fraction with its inverse, the total will be 34/15. Find this fraction**.**

4.66. The cost of production was 25 UAH. After two consecutive decreases by the same percent, it became 20.25 UAH. By what percentage did the cost price go down each time?

4.67. In the club hall, there are as many rows as there are seats in each row. If we increase the number of rows by 2 times and reduce the number of seats in each row by 10, the number of seats in the hall will increase by 300. How many rows are there in the hall?

4.68. The boatman travels a distance of 16 km along the river 6 hours faster than against the current. The speed of the boat in standing water is 2 km/h higher than the speed of the current. Determine the speed of the boat in still water and the speed of the current.

4.69. The tourist walked 105 km in a few days, covering the same distance every day. If he had spent 2 days more on this trip, he could have walked 6 km less per day. How many days did the journey last?

4.70. A boat and a raft set off from wharf A downstream to pier B. The boat reached pier B, turned back and met the raft in point C 4 hours after leaving A. How many hours did the boat sail from A to B?

4.71. The distance between the cities is 960 km. A passenger train passes this distance at a speed 20 km/h higher than the freight train. All the way, the passenger train runs 4 hours faster than the freight train. Find the speed of the trains.

4.72. A cyclist needs to ride a distance of 30 km. Having left 3 minutes before the appointed time, the cyclist was riding at a speed less than 1 km/h and arrived at the destination on time. How fast was the cyclist riding?

4.4. Equations of higher degrees

When solving equations above the second degree, two main methods are applied: factorization of the left side of the equation $F(x)=0$ and the introduction of a new variable. These methods are considered in the following examples.

Example 4.73. Solve the equation:

$$x^3-3x-2=0.$$

Solution: Factorize the left side of the equation using the grouping method. Then we have:

$$(x+1)\left(x^2-x-2\right)=0.$$

The task has been reduced to solving two simple equations:

$$\left[\begin{array}{l} x+1=0, \\ x^2-x-2=0, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=-1, \\ x=2. \end{array}\right. \Leftrightarrow \{-1;2\}.$$

Answer: $x=\{-1;2\}$

Example 4.74. Solve the equation:

$$x^3+x^2-x+15=0.$$

Solution: To factorize the left side of the equation, we find the roots of a polynomial. Consider the polynomial $P(x)=x^3+x^2-x+15$. As is known, if a polynomial with integer coefficients has integer roots, then they are divisors of the free term. The divisors of the free term are: $\{\pm1;\pm3;\pm5;\pm15\}$. Let us find the roots:

$$P(1)=1+1-1+15\neq0,\ P(-1)=-1+1+1+15\neq0,$$
$$P(3)=27+9-3+15\neq0,\ P(-3)=-27+9+3+15=0.$$

Then, $x=-3$ is the root. According to the Bezout theorem, the polynomial $P(x)$ is divided without residue by $(x+3)$. We can divide $P(x)$ according to the rule of polynomial division:

$$\begin{array}{r|l} x^3+x^2-x+15 & x+3 \\ \cline{2-2} -\ x^3+3x^2 \qquad\qquad & x^2-2x+5 \\ \cline{1-1} -2x^2-x+15 & \\ -\ -2x^2-6x \qquad & \\ \cline{1-1} 5x+15 & \\ -\ 5x+15 & \\ \cline{1-1} 0 & \end{array}$$

Thus, the left side of the equation decomposes into two factors:

$$(x+3)(x^2-2x+5)=0,$$

wherefrom we have:

$$x^2-2x+5=0.$$

The last equation has no solution because of $D<0$.

Answer: $x=\{-3\}$.

Example 4.75. Solve the equation:

$$(x-1)^3+(2x+3)^3=27x^3+8.$$

Solution: It is unlikely that this equation should lead to the previous type of equation by cubing the terms. Using the formula

$$a^3+b^3=(a+b)\cdot\left(a^2-ab+b^2\right)$$

we can transform the equation:

$$(x-1+2x+3)\cdot$$
$$\left((x-1)^2-(x-1)(2x+3)+(2x+3)^2\right)=(3x+2)(9x^2-6x+4),$$
$$(3x+2)(3x^2+9x+13)-(3x+2)(9x^2-6x+4)=0,$$
$$(3x+2)\cdot\left(-6x^2+15x+9\right)=0.$$

wherefrom: $$\left[\begin{array}{l}3x+2=0,\\ 6x^2-15x-9=0,\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x=-\dfrac{2}{3},\\ x=-\dfrac{1}{2},\\ x=3.\end{array}\right.$$

Answer: $x=\left\{-\dfrac{2}{3};-\dfrac{1}{2};3\right\}$.

Example 4.76. Solve the equation:

$$\left(x^2-2x-1\right)^2+3x^2-6x-13=0.$$

Solution: Transform the equation:

$$\left(x^2-2x-1\right)^2+3\left(x^2-2x\right)-13=0$$

and denote: $x^2-2x-1=y$. Then we have:

$$y^2+3(y+1)-13=0,$$

wherefrom we find: $y_1=-5$ and $y_2=2$.

So, the original equation is equivalent to the set of equations:

$$\left[\begin{array}{l} x^2-2x-1=-5, \\ x^2-2x-1=2, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x^2-2x+4=0, \\ x^2-2x-3=0. \end{array}\right.$$

The first equation has no solutions. The roots of the second equation are the numbers: $x_1=-1$ and $x_2=3$.

Answer: $x=\{-1;3\}$.

Example 4.77. Solve the equation:

$$(x-4)\cdot(x-5)\cdot(x-6)\cdot(x-7)=1680.$$

Solution: Rewrite the equation as

$$(x-4)\cdot(x-7)\cdot(x-5)\cdot(x-6)=1680$$

and multiply in this order by two brackets. Then we have:

$$\left(x^2-11x+28\right)\cdot\left(x^2-11x+30\right)=1680.$$

Using variable substitution, $x^2-11x+28=y$ we get $y^2+2y-1680=0$. The roots of this equation are the numbers: $y_1=-42$ and $y_2=40$.

By solving the set of equations

$$\left[\begin{array}{l} x^2-11x+28=-42, \\ x^2-11x+28=40, \end{array}\right.$$

we obtain from the second equation: $x_1=-1$ and $x_2=12$. The first equation has no real roots.

Answer: $x=\{-1;12\}$.

Example 4.78. Solve the equation:

$$\frac{x+1}{x^3+x-1}+\frac{x^3+x-1}{x+1}=2.$$

Solution: Let

$$\frac{x+1}{x^3+x-1}=y.$$

Then,

$$y+\frac{1}{y}=2,$$

wherefrom $y=1$ or $\frac{x+1}{x^3+x-1}=1 \Leftrightarrow x^3=2, x=\sqrt[3]{2}$.

Answer: $x=\left\{\sqrt[3]{2}\right\}$.

Example 4.79. Solve the equation:

$$4x^2+12x+\frac{12}{x}+\frac{4}{x^2}=47.$$

Solution: Group the terms:

$$4\left(x^2+\frac{1}{x^2}\right)+12\left(x+\frac{1}{x}\right)-47=0.$$

Let $x+\frac{1}{x}=y$, then $\left(x+\frac{1}{x}\right)^2=y^2$, wherefrom:

$$x^2+\frac{1}{x^2}=y^2-2.$$

Then, we have:

$$4\left(y^2-2\right)+12y-47=0,$$

wherefrom: $y_1=2{,}5$, $y_2=-5{,}5$.

So:

1) $x+\frac{1}{x}=2{,}5\Rightarrow x_1=2, x_2=\frac{1}{2};$

2) $x+\frac{1}{x}=-5{,}5;\ 2x^2+11x+2=0\Rightarrow x_{3,4}=\frac{-11\pm\sqrt{105}}{4}.$

Answer: $x=\left\{\frac{1}{2};2;\frac{-11\pm\sqrt{105}}{4}\right\}.$

Example 4.80. Solve the equation:

$$\frac{2x}{x^2-x+4}+\frac{7x}{x^2+3x+4}=\frac{11}{8}.$$

Solution: We divide the numerator and denominator of each fraction from the left by x. At the same time, there is no loss of the root, since the number x is not the root of this equation. As a result, we obtain the equation:

$$\frac{2}{x-1+\frac{4}{x}}+\frac{7}{x+3+\frac{4}{x}}=\frac{11}{8}.$$

Designating $x+\frac{4}{x}=y$, we have:

$$\frac{2}{y-1}+\frac{7}{y+3}=\frac{11}{8}.$$

Or, after transformations, we have:

$$\frac{-11y^2+50y+25}{(y-1)(y+3)}=0,$$

wherefrom, $y_1=5$ and $y_2=-5/11$.

Next, we have a set of two equations:

$$\left[\begin{array}{l} x+\frac{4}{x}=5, \\ x+\frac{4}{x}=-\frac{5}{11}, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x^2-5x+4=0, \\ 11x^2+5x+44=0, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=1, \\ x=4. \end{array}\right.$$

The second equation in this set of equations does not have real solutions.

Answer: $x=\{1;4\}$.

Example 4.81. Solve the equation:

$$4x^4-3x^3-2x^2-3x+4=0.$$

Solution: This equation is called the fourth-degree symmetric equation. In general, it looks as follows:

$$ax^4+bx^3+cx^2+bx+a=0,$$

where $a\neq 0$.

Dividing each term in the left side of this equation by x^2 ($x=0$ is not a root of this equation), we get the equivalent equation

$$4x^2-3x-2-\frac{3}{x}+\frac{4}{x^2}=0.$$

Let us rearrange the addends

$$4\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-2=0$$

and make a replacement: $x+\frac{1}{x}=y$.

Then, $x^2+\frac{1}{x^2}=y^2-2$ and the equation becomes as follows:

$$4\left(y^2-2\right)-3y-2=0 \Leftrightarrow 4y^2-3y-10=0.$$

From here we find: $y_1=-\frac{5}{4}$ and $y_2=2$. Returning to the original variable, we have:

$$\left[\begin{array}{l} x+\frac{1}{x}=2, \\ x+\frac{1}{x}=-\frac{5}{4}, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x^2-2x+1=0, \\ 4x^2+5x+4=0, \end{array}\right. \Leftrightarrow x=1.$$

Answer: $x=\{1\}$.

Example 4.82. Solve the equation: $x^2+\frac{4x^2}{(x+2)^2}=5$.

Solution: Select the square of the difference on the left, that is, transform the equation:

$$\left(x^2-2x\cdot\frac{2x}{x+2}+\frac{4x^2}{(x+2)^2}\right)+2x\cdot\frac{2x}{x+2}=5;$$

$$\left(x-\frac{2x}{x+2}\right)^2+\frac{4x^2}{(x+2)}=5;\ \left(\frac{x^2}{x+2}\right)^2+4\frac{x^2}{x+2}-5=0.$$

Let $\frac{x^2}{x+2} = y$, then $y^2 + 4y - 5 = 0$, wherefrom: $\left[\begin{array}{l} y_1 = -5, \\ y_2 = 1. \end{array}\right.$

Further:

1) $\frac{x^2}{x+2} = -5 \Leftrightarrow \begin{cases} x^2 + 5x + 10 = 0, \\ x \neq -2, \end{cases} \Leftrightarrow x \in \varnothing$;

2) $\frac{x^2}{x+2} = 1 \Leftrightarrow \begin{cases} x^2 - x - 2 = 0, \\ x \neq -2, \end{cases} \Leftrightarrow \left[\begin{array}{l} x = 2, \\ x = -1. \end{array}\right.$

Answer: $x = \{-1; 2\}$.

Example 4.83. Solve the equation:

$$\left(3x^2 + 7x - 2\right)^2 + 5x^2\left(3x^2 + 7x - 2\right) = 24x^4.$$

Solution: Divide both sides of the equation by x^4. As a result, we obtain the equivalent equation, because $x = 0$ is not the root of the equation. Then,

$$\left(\frac{3x^2 + 7x - 2}{x^2}\right)^2 + 5\frac{3x^2 + 7x - 2}{x^2} = 24.$$

After substitution $\frac{3x^2 + 7x - 2}{x^2} = t$, we have the square equation $t^2 + 5t - 24 = 0$ with roots: $t_1 = -8$ and $t_2 = 3$.

Moving to the original variable x, we obtain two equations:

$$\frac{3x^2 + 7x - 2}{x^2} = -8,$$

$$\frac{3x^2 + 7x - 2}{x^2} = 3,$$

with the solutions: $x=\frac{-7\pm\sqrt{137}}{22}$ or $x=\frac{2}{7}$.

Answer: $x=\left\{\left(-7\pm\sqrt{137}\right)/22;\ 2/7\right\}$.

In the considered examples of equations of higher degrees, various methods for their solution have been shown. The most common ways to replace a variable have been used. In some cases, such a replacement was made immediately, and in others – after targeted transformations.

Tasks for individual work

Solve the equations:

4.84. $(x-2)^6-19(x-2)^3=216$.

4.85. $27x^3-1=0$.

4.86. $\left(x^2-5x\right)^2+10\left(x^2-5x\right)+24=0$.

4.87. $\left(x^2+x+1\right)\cdot\left(x^2+x+2\right)=12$.

4.88. $\frac{4}{x^2+4}+\frac{5}{x^2+5}=2$.

4.89. $\frac{x^2+1}{x}+\frac{x}{x^2+1}=2{,}9$.

4.90. $x^4-27x=0$.

4.91. $x^4+x^3-12x^2=0$.

4.92. $x^4-3x^3+3x^2-x=0$.

4.93. $x^3-x^2-4x+4=0$.

4.94. $x^4+2x^3+x^2-4=0$.

4.95. $2x^3-5x^2-5x+2=0$.

4.96. $\left(x+\frac{8}{x}\right)^2+x=42-\frac{8}{x}$.

4.97. $\left(x^2-6x\right)^2-2(x-3)^2=81$.

4.98. $\left(x^2+x+1\right)^2-3x^2-3x-1=0$.

4.99. $\left(x^2-2x-4\right)\cdot\left(x^2-2x-3\right)=2$.

4.100. $(x+3)\cdot(x+1)\cdot(x+5)\cdot(x+7)=-16$.

4.101. $(6x+5)^2\cdot(3x+2)\cdot(x+1)=35$.

4.102. $\frac{2x}{3x^2-x+2}-\frac{7x}{3x^2+5x+2}=1$.

4.103. $\frac{x^2-10x+15}{x^2-6x+15}=\frac{4x}{x^2-12x+15}$.

4.104. $x^3+\frac{1}{x^3}=6\cdot\left(x+\frac{1}{x}\right)$.

4.105. $x^2+\frac{25x^2}{(x+1)^2}=11$.

4.106. $\left(\frac{x-2}{x}\right)^2+\left(\frac{x}{x-2}\right)^2=\frac{17}{4}$.

4.107. $\left(3x^2+7x-2\right)^2+5x^2\cdot\left(3x^2+7x-2\right)=24x^4$.

4.5. Irrational equations

An irrational equation is that one which has an unknown under the root symbol.

The basic methods for solving irrational equations are:

1) raising both parts of the equation to the same power;
2) introduction of a new variable.

The first method can lead to the appearance of extraneous roots; therefore, with this method of solution, it is necessary to check the roots by substituting them into the original equation. The latter is recommended in any case.

Example 4.108. Solve the equation:

$$\sqrt{x-12}+\sqrt{x-4}=-5.$$

Solution: All roots of an even degree are non-negative, therefore the left side of the equation cannot take a negative value for any valid value of x.

Answer: $x=\{\emptyset\}$.

Example 4.109. Solve the equation:

$$(x+4)\cdot\sqrt{2x-1}=0.$$

Solution: This equation is equivalent to the system:

$$\begin{cases}\left[\begin{array}{l}x+4=0,\\ \sqrt{2x-1}=0,\end{array}\right.\\ 2x-1\geq 0,\end{cases}\Leftrightarrow\begin{cases}\left[\begin{array}{l}x=-4,\\ x=0,5\end{array}\right.\\ x\geq 0,5,\end{cases}\Leftrightarrow x=0,5.$$

Answer: $x=\{0,5\}$.

Example 4.110. Solve the equation:

$$\sqrt{x-8}-\sqrt{5-x}=3.$$

Solution: Finding the domain of this equation will undoubtedly bring benefits, and it boils down to solving a system of linear inequalities

$$\begin{cases} x-8\geq 0, \\ 5-x\geq 0 \end{cases}$$

which have no solutions, therefore, the original equation has no solutions either.

Answer: $x = \{\emptyset\}$.

Example 4.111. Solve the equation:

$$x-\sqrt{x+1}=5.$$

Solution: Let us find the domain (D):

$$x+1\geq 0, x\geq -1, D=[-1;\infty).$$

Select the root and square both sides of the equation:

$$x-5=\sqrt{x-1}, \qquad (x-5)^2=\left(\sqrt{x+1}\right)^2,$$

$$x^2-10x+25=x+1, \quad x^2-11x+24=0,$$

wherefrom: $x_1=3, x_2=8.$

Although both roots belong to D, they should be checked. Verification shows that $x=3$ is the extraneous root.

Answer: $x=\{8\}$.

Example 4.112. Solve the equation:

$$\sqrt{2x-4}-\sqrt{x+5}=1.$$

Solution: Let us find the domain D:

$$\begin{cases} 2x-4\geq 0, \\ x+5\geq 0, \end{cases} \Leftrightarrow \begin{cases} x\geq 2, \\ x\geq -5, \end{cases} \Leftrightarrow x\geq 2, \quad D=[2;\infty).$$

Select one of the roots and square both sides of the equation:

$$\left(\sqrt{2x-4}\right)^2=\left(\sqrt{x+5}+1\right)^2; 2x-4=x+5+2\sqrt{x+5}+1;$$
$$2\sqrt{x+5}=x-10; \quad 4\cdot\left(x+5\right)=x^2-20x+100,$$

$$x^2-24x+80=0,$$

wherefrom: $x_1=4, x_2=20$. Both roots belong to the set D. The verification shows that $x=4$ is the extraneous root.

Answer: $x=\{20\}$.

Example 4.113. Solve the equation:

$$\sqrt{x+1}+\sqrt{4x+13}=\sqrt{3x+12}.$$

Solution: Let us find the domain D:

$$\begin{cases} x+1\geq 0, \\ 4x+13\geq 0, \\ 3x+12\geq 0, \end{cases} \Leftrightarrow \begin{cases} x\geq -1, \\ x\geq -\dfrac{13}{4}, \\ x\geq -4, \end{cases} \Leftrightarrow \ x\geq -1, \ D=\left[-1;\infty\right).$$

It is advisable to write down the equation in the form:

$$\sqrt{4x+13}=\sqrt{3x+12}-\sqrt{x+1}$$

and square both sides:

$$4x+13=3x+12-2\cdot\sqrt{\left(3x+12\right)\cdot\left(x+1\right)}+x+1;$$

$$2\cdot\sqrt{\left(3x+12\right)\cdot\left(x+1\right)}=0,$$

wherefrom: $x_1=-1, \ x_2=-4$.

The root $x=-4$ does not belong to D and then, this is an extraneous root. The verification shows that $x=-1$ satisfies this equation.

Answer: $x=\{-1\}$.

Example 4.114. Solve the equation:

$$\sqrt{x^3+4x-1-8\sqrt{x^4-x}}=\sqrt{x^3-1}+2\sqrt{x}.$$

Solution: Let us square both sides of the equation:

$$x^3+4x-1-8\cdot\sqrt{x^4-x}=x^3-1+4\cdot\sqrt{x^3-1}\cdot\sqrt{x}+4x.$$

After simplification, we have:

$$12\sqrt{\left(x^3-1\right)\cdot x}=0,$$

wherefrom: $x_1=0$; $x_2=1$. It is obvious that $x=0$ is an extraneous root.

Answer: $x=\{1\}$.

Example 4.115. Solve the equation:

$$\sqrt{x+\sqrt{6x-9}}=\sqrt{6}-\sqrt{x-\sqrt{6x-9}}.$$

Solution: Note the obvious limitation: $6x-9\geq 0$, $x\geq\frac{3}{2}$, which will help give the final answer. Let us transform the equation to the form:

$$\sqrt{x+\sqrt{6x-9}}+\sqrt{x-\sqrt{6x-9}}=\sqrt{6}$$

and square both sides:

$$x+\sqrt{6x-9}+2\cdot\sqrt{x^2-6x+9}+x-\sqrt{6x-9}=6,$$

$$2x+2\cdot\sqrt{(x-3)^2}=6,$$

$$\sqrt{(x-3)^2}=3-x,$$

$$|x-3|=3-x,$$

wherefrom: $x-3\leq 0$, that is, $x\leq 3$.

Considering that $x \geq \frac{3}{2}$, we have a set of solutions to the equation, which is determined by the inequality: $\frac{3}{2} \leq x \leq 3$.

Answer: $x \in \left[\frac{3}{2};3\right]$.

Example 4.116. Solve the equation: $\sqrt[3]{2x-1} + \sqrt[3]{x-1} = 1$.

Solution: Clearly, $x \in R$. Let us cube both sides of the equation using on the left the formula:

$$(a+b)^3 = a^3 + b^3 + 3ab \cdot (a+b),$$

$$2x-1+x-1+3 \cdot \sqrt[3]{2x-1} \cdot \sqrt[3]{x-1} \cdot \left(\sqrt[3]{2x-1} + \sqrt[3]{x-1}\right) = 1.$$

The expression in brackets by condition is equal to one, so the equation takes the form:

$$3x-2+3 \cdot \sqrt[3]{(2x-1) \cdot (x-1)} = 1.$$

Let us transform this equation to the form:

$$\sqrt[3]{(2x-1) \cdot (x-1)} = 1-x$$

and then cube both sides. We have:

$$(2x-1) \cdot (x-1) = (1-x)^3$$

or, after transformations: $x^3 - x^2 = 0$, wherefrom: $x_1 = 0$, $x_2 = 1$.

It should be noted that when solving the equation in this way, it is necessary to make sure there are no extraneous roots among the roots obtained. Indeed, verification shows that the root $x = 0$ does not satisfy the original equation.

Answer: $x = \{1\}$.

The following examples apply the introduction of a new unknown – one of the most important methods for solving nonlinear equations. Using a new variable, one can reduce the equation to a rational or irrational equation of the standard form. After the introduction of a new variable, the resulting equation needs to be solved and extraneous roots, if they have appeared, have to be discarded and only then you should return to the original variable.

Example 4.117. Solve the equation: $x\cdot\sqrt{x^2+15}-\sqrt{x}\cdot\sqrt[4]{x^2+15}=2$.

Solution: We have the domain: $D=[0;\infty)$. Let us write down the equation in the form

$$\sqrt{x^2\cdot\left(x^2+15\right)}-\sqrt[4]{x^2\cdot\left(x^2+15\right)}=2$$

and make the replacement:

$$\sqrt[4]{x^2\cdot\left(x^2+15\right)}=t,\ t\geq 0.$$

After that we get the equation for t:

$$t^2-t-2=0,\ t_1=2;\ t_2=-1.$$

But $t_2=-1$ does not satisfy the requirement $t\geq 0$. The root $t_1=2$ remains. Let us go back to the variable x:

$$\sqrt[4]{x^{2\cdot}\left(x^2+15\right)}=t_1=2,$$

$$x^4+15x^2=16,$$

$$x^4+15x^2-16=0.$$

This biquadratic equation has a solution $x^2=1$, wherefrom the solution is $x=1$ because $x\geq 0$ due to the domain and that is why the root $x=-1$ is extraneous.

Answer: $x=\{1\}$.

Example 4.118. Solve the equation:

$$3x^2+15x+2\cdot\sqrt{x^2+5x+1}=2.$$

Solution: Let us transform the equation to the form:

$$3\cdot\left(x^2+15x+1-1\right)+2\sqrt{x^2+5x+1}-2=0$$

and make the replacement:

$$\sqrt{x^2+x+1}=t,\ t\geq 0.$$

We get the quadratic equation for t:

$$3\cdot\left(t^2-1\right)+2t-2=0.$$

That is, $3t^2+2t-5=0$, wherefrom; $t_1=1$, $t_2=-\frac{5}{3}<0$. The root t_2 is not valid. Then we have a simple irrational equation for x:

$$\sqrt{x^2+x+1}=1,\ x^2+x=0,\ x_1=0,\ x_2=-1.$$

Answer: $x=\{-1;0\}$.

Example 4.119. Solve the equation:

$$\sqrt{x^2-4x+20}+\sqrt{x^2-4x+11}=9.$$

Solution: With the aid of the new variable $t=x^2-4x+11$ the equation is transformed to an irrational equation of the standard type:

$$\sqrt{t+9}+\sqrt{t}=9,\ t\geq 0.$$

Let us solve it:

$$\sqrt{t+9}=9-\sqrt{t}, \quad t+9=81-18\cdot\sqrt{t}+t, \quad \sqrt{t}=4, \ t=16.$$

The root $t=16$ satisfies the equation:

$$\sqrt{t+9}+\sqrt{t}=9.$$

Let us now return to the original variable:

$$x^2-4x+11=16, \ x^2-4x-5=0, \ x_1=5, \ x_2=-1.$$

Answer: $x=\{-1;5\}$.

Example 4.120. Solve the equation:

$$\sqrt{x^2-4x+3}+\sqrt{-x^2+3x-2}=\sqrt{x^2-x}.$$

Solution: Finding the set of acceptable values will greatly simplify the solution of this equation. Indeed, for this one we have to solve a system of three square inequalities by the method of intervals (Fig. 4.3):

$$\begin{cases} x^2-4x+3\ge 0, \\ -x^2+3x-2\ge 0, \\ x^2-x\ge 0, \end{cases} \qquad \begin{cases} (x-1)\cdot(x-3)\ge 0, \\ (x-1)\cdot(x-2)\le 0, \\ x\cdot(x-1)\ge 0, \end{cases} \qquad \begin{cases} \left[\begin{array}{l} x\le 1, \\ x\ge 3, \end{array}\right. \\ 1\le x\le 2, \\ \left[\begin{array}{l} x\le 0, \\ x\ge 1. \end{array}\right. \end{cases}$$

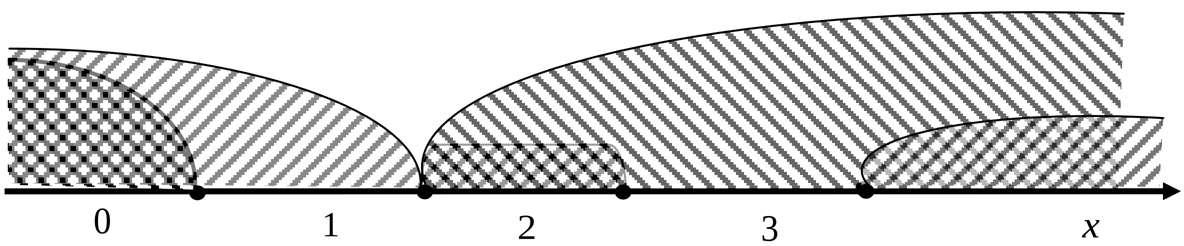


Fig. 4.3. **The solution of the equation by the method of intervals**

All three inequalities are simultaneously fulfilled with $x=1$. Then, the domain consists of one value $x=1$. Verification shows that this value is the root of the equation.

Answer: $x=\{1\}$.

Example 4.121. Solve the equation:

$$\sqrt{x+8+2\cdot\sqrt{x+7}}+\sqrt{x+1-\sqrt{x+7}}=4.$$

Solution: We have a limitation: $x\geq -7$. Let $\sqrt{x+7}=t \quad (t\geq 0)$. Then, $x+7=t^2$, $x=t^2-7$.

For the new variable t we get the equation:

$$\sqrt{t^2-7+8+2t}+\sqrt{t^2-7+1-t}=4,$$

or,

$$\sqrt{(t+1)^2}+\sqrt{t^2-t-6}=4,$$

which is equivalent to

$$|t+1|+\sqrt{t^2-t-6}=4.$$

Because $t+1>0$, then $|t+1|=t+1$ and the equation takes the form:

$$\sqrt{t^2-t-6}=3-t.$$

The solution of this equation can be reduced to an equivalent system:

$$\begin{cases} t^2-t-6=(3-t)^2, \\ 3-t\geq 0, \\ t\geq 0, \end{cases} \Leftrightarrow \begin{cases} t^2-t-6=9-6t+t^2 \\ 0\leq t\leq 3. \end{cases}.$$

From this: $t=3$. Then, $\sqrt{x+7}=3$, and the solution is: $x=9-7=2$.

Answer: $x=\{2\}$.

Let's consider some more useful examples in which some non-standard methods of solution will be applied.

Example 4.122. Solve the equation:

$$\frac{\sqrt{3-x}+\sqrt{x-2}}{\sqrt{3-x}-\sqrt{x-2}}=\frac{1}{5-2x}.$$

Solution: Obviously, the domain is:

$$D:\begin{cases}3-x\geq 0,\\ x-2\geq 0,\\ 5-2x\neq 0,\end{cases} \text{ that is, } x\in\left[2;\frac{5}{2}\right)\cup\left(\frac{5}{2};3\right].$$

Let's get rid of irrationality in the denominator on the left side of the equation:

$$\frac{\left(\sqrt{3-x}+\sqrt{x-2}\right)\left(\sqrt{3-x}+\sqrt{x-2}\right)}{\left(\sqrt{3-x}-\sqrt{x-2}\right)\left(\sqrt{3-x}+\sqrt{x-2}\right)}=\frac{1}{5-2x}.$$

Then:

$$\frac{\left(\sqrt{3-x}+\sqrt{x-2}\right)^2}{3-x-x+2}=\frac{1}{5-2x};\quad \frac{1+2\sqrt{(3-x)(x-2)}}{5-2x}=\frac{1}{5-2x},$$

$$1+2\sqrt{(3-x)(x-2)}=1,\ \sqrt{(3-x)(x-2)}=0,\ x_1=3,\ x_2=2.$$

Answer: $x=\{2;3\}$.

Example 4.123. Solve the equation:

$$\frac{\sqrt[7]{x-\sqrt{2}}}{2}-\frac{\sqrt[7]{x-\sqrt{2}}}{x^2}=\frac{x}{2}\sqrt[7]{\frac{x^3}{x+\sqrt{2}}}.$$

Solution: It is obvious that: $x \neq 0, x \neq -\sqrt{2}$ (the domain).
Using simple calculations, we transform this seemingly complicated equation:

$$\sqrt[7]{x-\sqrt{2}}\left(\frac{1}{2}-\frac{1}{x^2}\right)=\frac{x}{2}\frac{\sqrt[7]{x^3}}{\sqrt[7]{x+\sqrt{2}}}, \quad \sqrt[7]{x^2-2}\cdot\frac{x^2-2}{2x^2}=\frac{x\sqrt[7]{x^3}}{2},$$

$$\sqrt[7]{x^2-2}\cdot\left(x^2-2\right)=x^3\sqrt[7]{x^3}.$$

To get rid of irrationality, we will raise both sides of the equation to the seventh power:

$$\left(x^2-2\right)\left(x^2-2\right)^7=x^{21}x^3 \Leftrightarrow \left(x^2-2\right)^8=x^{24}.$$

Taking the eighth degree root from both sides of the equation, we get:

$$\left|x^2-2\right|=\left|x^3\right|.$$

This equation is equivalent to the set of equations:

$$\left[\begin{array}{l} x^2-2=x^3, \\ x^2-2=-x^3; \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x^3-x^2+2=0, \\ x^3+x^2-2=0; \end{array}\right. \Leftrightarrow$$

$$\left[\begin{array}{l} \left(x^3+1\right)-\left(x^2-1\right)=0, \\ \left(x^3-1\right)+\left(x^2-1\right)=0; \end{array}\right. \Leftrightarrow \left[\begin{array}{l} (x+1)\left(x^2-x+1-x+1\right)=0, \\ (x-1)\left(x^2+x+1+x+1\right)=0; \end{array}\right. \Leftrightarrow$$

$$\left[\begin{array}{l} (x+1)\left(x^2-2x+2\right)=0, \\ (x-1)\left(x^2+2x+2\right)=0; \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x+1=0, \\ x-1=0, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=-1, \\ x=1. \end{array}\right.$$

Answer: $x=\{-1;1\}$.

Tasks for individual work

Solve the equations:

4.124. $\sqrt{5+2x}=5-x$. **4.125.** $\sqrt{6-4x-x^2}=x+4$.

4.126. $\left(9-x^2\right)\sqrt{2-x}=0$. **4.127.** $\dfrac{\sqrt{5-x^2}}{x+1}=1$. **4.128.** $\dfrac{2+\sqrt{19-2x}}{x}=1$.

4.129. $\sqrt[3]{16-x^3}=4-x$. **4.130.** $\sqrt[5]{25+\sqrt{x-4}}=2$. **4.131.** $\sqrt{x+5}-\sqrt{x}=1$.

4.132. $\sqrt{2x-4}-\sqrt{x+5}=1$. **4.133.** $\sqrt{4-x}+\sqrt{5+x}=3$.

4.134. $\sqrt{2x+3}+\sqrt{3x-1}=\sqrt{5x+2}$. **4.135.** $\sqrt{9-5x}=\sqrt{3-x}+\dfrac{6}{\sqrt{3-x}}$.

4.136. $\sqrt[3]{x^2}-\sqrt[3]{x}-6=0$. **4.137.** $\dfrac{4}{\sqrt[3]{x}+2}+\dfrac{\sqrt[3]{x}+3}{5}=2$.

4.138. $\sqrt{2-x}+\dfrac{4}{\sqrt{2-x}+3}=2$. **4.139.** $x\sqrt[3]{x}-4\sqrt[3]{x^2}+4=0$.

4.140. $3\sqrt[3]{x}-5\sqrt[3]{x^{-1}}=2x^{-1}$. **4.141.** $2\sqrt[3]{x}+\sqrt[6]{x}=10$. **4.142.** $\dfrac{\sqrt{x}+\sqrt[3]{x}}{\sqrt{x}-\sqrt[3]{x}}=3$.

4.143. $\sqrt{x^2+32}-2\sqrt[4]{x^2+32}=3$. **4.144.** $\sqrt{x+5}+\sqrt[4]{x+5}=12$.

4.145. $x^2+2x+\sqrt{x^2+2x+8}=12$.

4.146. $\sqrt{3x^2-2x+15}+\sqrt{3x^2-2x+8}=7$. **4.147.** $\sqrt[3]{8x+4}-\sqrt[3]{8x-4}=2$.

4.148. $3\sqrt{x+1}+|x-5|=6$. **4.149.** $4\sqrt{x+2}=|x+1|+4$.

4.150. $\sqrt{1+x\sqrt{x^2-24}}=x-1$. **4.151.** $\sqrt[3]{2x+3}+\sqrt[3]{3x-1}=\sqrt[3]{5x+2}$.

4.152. $\sqrt[3]{2+11x}+\sqrt[3]{2-11x}=4$.

4.153. $\sqrt{x^2-4x+4}-\sqrt{x^2-6x+9}=\sqrt{x^2-2x+1}$.

4.154. $\sqrt{x+5-4\sqrt{x+1}}+\sqrt{x+2-2\sqrt{x+1}}=1$.

4.155. $\sqrt{x+2\sqrt{x-1}}-\sqrt{x-2\sqrt{x-1}}=2$. **4.156.** $\dfrac{x\sqrt[3]{x}-1}{\sqrt[3]{x^2}-1}-\dfrac{\sqrt[3]{x^2}-1}{\sqrt[3]{x}+1}=4$.

4.157. $\dfrac{4}{x+\sqrt{x^2+x}}-\dfrac{1}{x-\sqrt{x^2+x}}=\dfrac{3}{x}$.

Questions for self-assessment

1. What is a linear algebraic equation?
2. What is a linear algebraic equation with parameters?
3. What is a quadratic equation?
4. What is the equation of higher degree?
5. What are the irrational equations?

5. Systems of algebraic equations

5.1. Systems of linear equations with two unknowns

A system of equations is called linear if the equations in the system are linear.

The system of two linear equations with two unknowns is as follows:

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases}$$

The solution of a system of equations is an ordered pair of numbers, which is a solution to each of the equations in the system.

When solving the system there are three possible cases:

1) the system has an only solution if

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2};$$

2) the system has countless solutions if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2};$$

3) the system has no solutions if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

The basic methods for solving systems of linear equations

The following methods are used for solving systems of linear equations:

1) the substitution method;
2) the method of algebraic addition;
3) Cramer's formula.

The substitution method consists in expressing one variable from some equation through another variable and substituting it into another equation. The result is an equation with one unknown, from which this unknown can be found with further finding the remaining unknowns.

Example 5.1. Solve the system:

$$\begin{cases} 3x+4y=11, \\ 5x-3y=-1. \end{cases}$$

Solution: In the first equation we express x through y:

$$x=\frac{11-4y}{3}.$$

The found value of x is substituted into the second equation:

$$5\frac{11-4y}{3}-3y=-1.$$

Multiplying by 3, we have:

$$55-20y-9y=-3 \text{ or } 29y=58,$$

wherefrom: $y=2$.

Now we can find x:

$$x=\frac{11-4\cdot 2}{3},$$

wherefrom: $x=1$. Then, $\{(1;2)\}$ is the solution of the given equation.

The method of algebraic addition implies that with the help of transformations the equations turn to an equivalent system in which one of the equations contains only one variable.

Example 5.2. Solve the system:

$$\begin{cases} 2x - 3y = 8, \\ 7x - 5y = -5. \end{cases}$$

Solution: Let us multiply the first equation by 7, and the second by (−2). Then we get:

$$\begin{cases} 14x - 21y = 56, \\ -14x + 10y = 10. \end{cases}$$

Adding these equations, we get $-11y = 66$ wherefrom: $y = -6$. The obtained $y = -6$ is substituted, for example, in the first equation of the system. Then, $2x - 3(-6) = 8$. Wherefrom: $x = -5$.

So, the set $\{(-5; -6)\}$ is a solution to the original system of equations.

Example 5.3. Solve the system

$$\begin{cases} x + y = 4, \\ |2x - y| = 5. \end{cases}$$

Solution: Taking into account the definition of the module, the system splits into two systems:

1) $\begin{cases} x + y = 4, \\ 2x - y = 5; \\ 2x - y \geq 0. \end{cases} \Rightarrow \begin{cases} x + y = 4, \\ 3x = 9; \\ 2x > y. \end{cases} \Rightarrow \begin{cases} y = 1, \\ x = 3; \\ 2x > y. \end{cases} \Rightarrow \begin{cases} x = 3, \\ y = 1. \end{cases}$

2) $\begin{cases} x + y = 4, \\ 2x - y = -5; \\ 2x - y < 0. \end{cases} \Rightarrow \begin{cases} x + y = 4, \\ 3x = -1; \\ 2x < y. \end{cases} \Rightarrow \begin{cases} y = 13/3, \\ x = -1/3; \\ 2x < y. \end{cases} \Rightarrow \begin{cases} x = -1/3, \\ y = 13/3. \end{cases}$

Example 5.4. Solve the system:

$$\begin{cases} 6ax - y = 7, \\ -ax + y = 2. \end{cases}$$

Solution: This system contains the parameter a. Adding these two equations, we get:

$$5ax = 9,$$

wherefrom: $x = \dfrac{9}{5a}$, if $a \neq 0$.

From the second equation we find y, knowing x:

$$y = 2 + a \cdot \frac{9}{5a}, \quad y = \frac{19}{5}, \text{ if } a \neq 0.$$

If $a = 0$, the system has no solutions.

So, if $a \neq 0$, $x = \dfrac{9}{5a}$, $y = \dfrac{19}{5}$; if $a = 0$, there is no solution.

Example 5.5. For what values of m does the system of equations

$$\begin{cases} (m+1)x - my = 4 \\ 3x - 5y = m \end{cases}$$

have such solutions for which $x - y < 2$?

Solution: This is a system of linear equations with a parameter *m*. Let us exclude y from the system. To do this, multiply the first equation by 5, and the second one by $(-m)$:

$$\begin{cases} (5m+5)x - 5my = 20; \\ -3mx + 5my = -m^2. \end{cases}$$

Adding these two equations, we have:

$$(5m+5-3m)x = 20-m^2 \ ,$$

wherefrom:

$$x = \frac{20-m^2}{2m+5}.$$

The value of y can be found by substituting the found value of x into the second equation of the original system:

$$\begin{cases} x = \dfrac{20-m^2}{2m+5}, \\ 3 \cdot \dfrac{20-m^2}{2m+5} - 5y = m, \end{cases} \Rightarrow \begin{cases} x = \dfrac{20-m^2}{2m+5}, \\ \dfrac{60-3m^2-2m^2-5m}{2m+5} = 5y. \end{cases}$$

Next we have:

$$\begin{cases} x = \dfrac{20-m^2}{2m+5}, \\ y = -\dfrac{m^2+m-12}{2m+5}. \end{cases}$$

Let us find m from the condition $x-y<2$:

$$\frac{20-m^2}{2m+5} + \frac{m^2+m-12}{2m+5} < 2 \Rightarrow$$

$$\frac{20-m^2+m^2+m-12}{2m+5} < 2 \quad \Rightarrow$$

$$\frac{m+8}{2m+5} - 2 < 0 \quad \Rightarrow \quad \frac{-3m-2}{2m+5} < 0 \quad \Rightarrow \quad \frac{3m+2}{2m+5} > 0.$$

We can apply the interval method to solve the last inequality (Fig.).

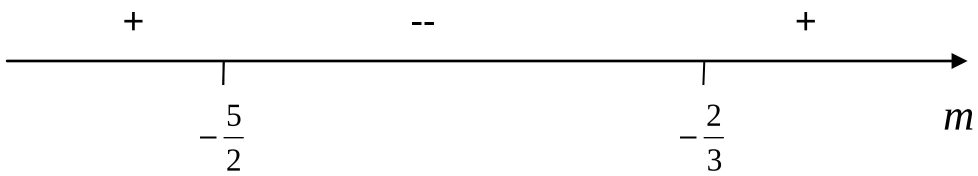


Fig. **The intervals**

It is clear that the solution to the inequality $x-y<2$ is the union of two intervals:

$$m\in\left(-\infty;-\frac{5}{2}\right)\cup\left(-\frac{2}{3};+\infty\right).$$

5.2. Systems of equations resulting in a linear equation

Some systems of two equations with two unknowns that are not linear are reduced to linear equations with respect to new variables by changing the variables.

Example 5.6. Solve the system:

$$\begin{cases}\dfrac{4}{x+y-1}-\dfrac{5}{2x-y+3}=-\dfrac{5}{2},\\ \dfrac{3}{x+y-1}+\dfrac{1}{2x-y+3}=-\dfrac{7}{5}.\end{cases}$$

Solution: This system is non-linear with respect to variables x and y.

We introduce new variables: $u=\dfrac{1}{x+y-1}$; $\quad v=\dfrac{1}{2x-y+3}$.

The system will take the form:

$$4u-5v=-\frac{5}{2},$$

$$3u+v=-\frac{7}{5}.$$

This system is linear regarding the new variables. Multiplying the second equation by 5 and adding it to the first one, we have

$$19u = -\frac{19}{2},$$

wherefrom: $u = -\frac{1}{2}$.

Substituting the obtained u in the second equation, we find v:

$$v = \frac{1}{10}.$$

Returning to the variables x and y, we have:

$$\begin{cases} \dfrac{1}{x+y-1} = -\dfrac{1}{2}, \\ \dfrac{3}{2x-y+3} = \dfrac{1}{10}. \end{cases} \Rightarrow \quad \begin{cases} x+y-1 = -2, \\ 2x-y+3 = 10. \end{cases} \Rightarrow$$

$$\begin{cases} x+y = -1, \\ 2x-y = 7. \end{cases} \Rightarrow \quad \begin{cases} x = 2, \\ y = -3. \end{cases}$$

The solution of the system is: $\{(2;-3)\}$.

5.3. Second order determinants

The determinant of the second order, composed of the numbers a_1, b_1, a_2, b_2, is the number d defined by the equality:

$$d = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 \cdot b_2 - a_2 \cdot b_1.$$

The numbers a_1, b_1, a_2, b_2 are called determinant elements; the elements a_1, b_2 form the forward diagonal, the elements b_1, a_2 form the backward diagonal.

Example 5.7. Compute the determinant:

$$\begin{vmatrix} 2 & 8 \\ -4 & 5 \end{vmatrix} = 2 \cdot 5 - (-4) \cdot 8 = 42.$$

5.4. Cramer's formula

A system of two linear equations with two unknowns is given:

$$\begin{cases} a_1 x + b_1 y = c_1, \\ a_2 x + b_2 y = c_2. \end{cases}$$

If the determinant of the system is nonzero,

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

the system has a unique solution, which can be found by the Cramer's formula:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}.$$

Here, Δ is the main determinant of the system, that is

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

Δ_x and Δ_y are the determinants obtained from the determinant Δ by replacing the column of coefficients for the corresponding variable by the column of free terms, i.e.

$$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

If, on the contrary, $\Delta = 0$, then the system either has no solution (when $\Delta_x \neq 0$ and $\Delta_y \neq 0$), or has countless solutions (when $\Delta_x = 0$ and $\Delta_y = 0$).

Example 5.8. Solve the system:

$$\begin{cases} 5x - 4y = 22, \\ 2x + 3y = -5. \end{cases}$$

Solution: We use the Cramer's formula:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}.$$

We compute further the determinant:

$$\Delta = \begin{vmatrix} 5 & -4 \\ 2 & 3 \end{vmatrix} = 15 + 8 = 23,$$

$$\Delta_x = \begin{vmatrix} 22 & -4 \\ -5 & 3 \end{vmatrix} = 66 - 20 = 46; \qquad \Delta_y = \begin{vmatrix} 5 & 22 \\ 2 & -5 \end{vmatrix} = -25 - 44 = -69.$$

Then: $x = \frac{46}{23} = 2; \quad y = -\frac{69}{23}.$

The system solution is: $\{(2;-3)\}$.

Example 5.9. Solve the system:

$$\begin{cases} ax + 3y = a, \\ 3x + ay = a. \end{cases}$$

Solution: The system determinant is:

$$\Delta = \begin{vmatrix} a & 3 \\ 3 & a \end{vmatrix} = a^2 - 9.$$

If $a^2 - 9 \neq 0$, that is $a \neq \pm 3$, the system has a unique solution:

$$x = \frac{\begin{vmatrix} a & 3 \\ a & a \end{vmatrix}}{a^2 - 9} = \frac{a^2 - 3a}{a^2 - 9} = \frac{a}{a+3}; \qquad y = \frac{\begin{vmatrix} a & a \\ 3 & a \end{vmatrix}}{a^2 - 9} = \frac{a^2 - 3a}{a^2 - 9} = \frac{a}{a+3}.$$

For $a=3$ the determinant of the system is zero, and the system has the form:

$$\begin{cases} 3x+3y=3, \\ 3x+3y=3. \end{cases}$$

It follows from this that the system of equations has an innumerable set of solutions. We set x arbitrarily, then $y=1-x$.

For $a=-3$ the determinant of the system is zero too, but the system has the form:

$$\begin{cases} -3x+3y=-3, \\ 3x-3y=-3. \end{cases} \qquad \begin{cases} x-y=1, \\ x-y=-1. \end{cases}$$

It follows from this that the system of equations has no solutions (the system is contradictory).

5.5. Solution of systems of linear equations by the Gauss method

Example 5.10. Solve the system:

$$\begin{cases} x_1-2x_2+3x_3=5, \\ 2x_1-3x_2-4x_3=8, \\ 4x_1-2x_2+5x_3=8. \end{cases}$$

Solution: To solve this system, we use the method of consistent elimination of unknowns (the Gauss method), with the help of which the system is reduced to a triangular form.

First, we exclude x_1 from the second and third equations. For this purpose, we leave the first equation unchanged multiplying both its parts by (-2) and add it to the second equation, then multiply both parts of the first equation by (-4) and add it to the third equation. Then the system takes the following form:

$$\begin{cases} x_1-2x_2+3x_3=5, \\ \qquad x_2-10x_3=-2, \\ \qquad 6x_2-7x_3=-12. \end{cases}$$

Now, exclude x_2 from the third equation. To do this, we multiply both sides of the second equation by (-6) and add it to the third equation. The system takes the form:

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 5, \\ \qquad x_2 - 10x_3 = -2, \\ \qquad\qquad 53x_3 = 0. \end{cases}$$

So, the system is reduced to a triangular form. From the last equation we find $x_3 = 0$.

Substituting it into the second equation, we find $x_2 = -2$. Knowing x_2 and x_3, substitute them into the first equation and find: $x_1 = 5 + 2 \cdot (-2)$, $x_1 = 1$.

So, the solution is: $\{(1; -2; 0)\}$.

5.6. The non-linear systems

Example 5.11. Solve the system:

$$\begin{cases} x - y = 2, \\ x^2 - xy + y^2 = 7. \end{cases}$$

Solution: We will solve the system by the substitution method. From the first equation, we express x through y and substitute it into the second equation. Then, we obtain:

$$\begin{cases} x = 2 + y, \\ (2+y)^2 - (2+y)y + y^2 = 7, \end{cases} \quad \text{or:} \quad \begin{cases} x = 2 + y, \\ y^2 + 2y - 3 = 0. \end{cases}$$

Solving the second equation, we have:

$$y = -1 \pm \sqrt{1+3}, \quad y_1 = -3, \quad y_2 = 1.$$

From the first equation, we have: $x_1 = -1$, $y_2 = 3$. So, the solution of the system is as follows: $\{(-1; -3), (3; 1)\}$.

Example 5.12. Solve the system:

$$\begin{cases} \sqrt{\dfrac{3x-2y}{2x}} + \sqrt{\dfrac{2x}{3x-2y}} = 2, \\ x^2 - 8 = 2x(y-3). \end{cases}$$

Solution: In the first equation we introduce the substitution:

$$\sqrt{\frac{3x-2y}{2x}} = t.$$

Then, the equation takes the form:

$$t + \frac{1}{t} = 2.$$

This equation is reduced to a quadratic equation:

$$t^2 - 2t + 1 = 0,$$

wherefrom: $t_1 = t_2 = 1$. Then,

$$\frac{3x-2y}{2x} = 1,$$

wherefrom: $x = 2y$. The system takes the form:

$$\begin{cases} x = 2y, \\ x^2 - 8 = 2x(y-3), \end{cases} \begin{cases} x = 2y, \\ (2y)^2 - 8 = 2 \cdot 2y(y-3), \end{cases} \Rightarrow \begin{cases} x = 2y, \\ -12y = -8, \end{cases}$$

$$\Rightarrow \begin{cases} x = \dfrac{4}{3}, \\ y = \dfrac{2}{3}. \end{cases}$$

So, the solution is: $\left\{\left(\dfrac{4}{3}; \dfrac{2}{3}\right)\right\}$.

5.7. Homogeneous systems of equations

If the left sides of the equations are homogeneous polynomials, then the system is called homogeneous. It is solved using the algebraic addition and the introduction of new variables.

Example 5.13. Solve the system:

$$\begin{cases} x^2 + 3xy + 2y^2 = 6, \\ 3x^2 - xy - y^2 = 1. \end{cases}$$

Solution: Let us multiply the second equation by (-6) and add it to the first equation:

$$-17x^2 + 9xy + 8y^2 = 0.$$

From the system we can see that $y \neq 0$. (For $y = 0$ we have:

$$\begin{cases} x^2 = 6, \\ 3x^2 = 1, \end{cases}$$

then, no value $x \in R$ can satisfy this system).

Dividing both sides of the last equation by $(-y^2)$, we have:

$$17\frac{x^2}{y^2} - 9\frac{x}{y} - 8 = 0.$$

Let us introduce a new variable: $\frac{x}{y} = t$. Then the equation takes the form:

$$17 \cdot t^2 - 9 \cdot t - 8 = 0.$$

We find the solution:

$$t_{12} = \frac{9 \pm \sqrt{81 + 544}}{34} \quad t_1 = -\frac{8}{17}, \quad t_2 = 1.$$

Then, $\frac{x}{y}=-\frac{8}{17}$ or $\frac{x}{y}=1.$

So, the original system is equivalent to the following two systems:

$$\left[\begin{array}{l}\begin{cases}x=-\frac{8}{17}y,\\ 3x^2-xy-y^2=1,\end{cases}\\ \begin{cases}x=y\\ 3x^2-xy-y^2=1.\end{cases}\end{array}\right.$$

Solving the first system, we have:

$$\begin{cases}x=-\frac{8}{17}y,\\ 3\frac{64}{289}y^2-\frac{8}{17}y^2-y^2=1.\end{cases}\Rightarrow\begin{cases}x=-\frac{8}{17}y,\\ y=\pm\frac{17}{\sqrt{39}}.\end{cases}$$

So, the solution of the first system is:

$$x_1=-\frac{8}{\sqrt{39}},y_1=\frac{17}{\sqrt{39}},x_2=\frac{8}{\sqrt{39}},y_2=-\frac{17}{\sqrt{39}}.$$

Solving the second system, we have:

$$\begin{cases}x=y,\\ 3y^2-y^2-y^2=1.\end{cases}\Rightarrow\begin{cases}x=y,\\ y=\pm 1.\end{cases}$$

The solution of this system is: $x_3=y_3=-1; x_4=y_4=1.$

So, this system has the following solutions:

$$\left(-\frac{8}{\sqrt{39}},\frac{17}{\sqrt{39}}\right);\left(\frac{8}{\sqrt{39}},-\frac{17}{\sqrt{39}}\right);(-1,-1);(1,1).$$

5.8. Symmetric systems of equations

A system is called symmetric if all its equations are symmetric, i.e. when x is replaced by y and y by x, it does not change. Such systems are solved by changing variables.

Example 5.14. Solve the system:

$$\begin{cases} xy + x + y = 5, \\ xy^2 + x^2 y = 6, \end{cases} \Rightarrow \begin{cases} xy + x + y = 5, \\ xy(y + x) = 6. \end{cases}$$

This system is symmetric. Let us set new variables: $x + y = u, \quad xy = v.$ Then the system takes the form:

$$\begin{cases} v + u = 5 \\ v \cdot u = 6 \end{cases}.$$

We solve this system by the substitute method:

$$\begin{cases} v + u = 5, \\ v \cdot u = 6, \end{cases} \Rightarrow \begin{cases} u = 5 - v, \\ v \cdot (5 - v) = 6, \end{cases} \Rightarrow \begin{cases} u = 5 - v, \\ v^2 - 5v + 6 = 0, \end{cases}$$

$$\begin{cases} u = 5 - v, \\ v_1 = 2, v_2 = 3, \end{cases} \Rightarrow \begin{cases} u_1 = 3, u_2 = 2, \\ v_1 = 2, v_2 = 3. \end{cases}$$

Go back to the former variables. Then we have two systems:

$$1) \begin{cases} x + y = 3, \\ xy = 2, \end{cases} \quad 2) \begin{cases} x + y = 2, \\ xy = 3. \end{cases}$$

Let us solve the first system:

$$\begin{cases} x + y = 3, \\ xy = 2, \end{cases} \Rightarrow \begin{cases} x = 3 - y, \\ (3 - y)y = 2, \end{cases} \Rightarrow \begin{cases} x = 3 - y, \\ y^2 - 3y + 2 = 0, \end{cases}$$

$$\begin{cases} x_1 = 2, x_2 = 1, \\ y_1 = 1, y_2 = 2. \end{cases}$$

Let us solve the second system:

$$\begin{cases} x+y=2, \\ xy=3, \end{cases} \Rightarrow \begin{cases} x=2-y, \\ (2-y)y=3, \end{cases} \Rightarrow \begin{cases} x=2-y, \\ y^2-2y+3=0. \end{cases}$$

Since the second equation has a negative discriminant $D=4-4\cdot 3=-8$, it has no real roots.

So, the solution of the system is as follows: $\{(2,1);(1,2)\}$.

Example 5.15. Solve the system:

$$\begin{cases} xy(x+y)=30, \\ x^3+y^3=35. \end{cases}$$

Solution: In the second equation we use the sum of the cubes formula, so we have:

$$\begin{cases} xy(x+y)=30, \\ (x+y)(x^2-xy+y^2)=35, \end{cases} \Rightarrow \begin{cases} xy(x+y)=30, \\ (x+y)((x+y)^2-3xy)=35. \end{cases} \Rightarrow$$

Now, let's change the variables: $x+y=u, \quad xy=v$. The system takes the form:

$$\begin{cases} vu=30, \\ u(u^2-3v)=35, \end{cases} \Rightarrow \begin{cases} vu=30, \\ u^3-3uv=35, \end{cases} \Rightarrow \begin{cases} vu=30, \\ u^3-90=35, \end{cases} \Rightarrow \begin{cases} v=6, \\ u=5. \end{cases}$$

We go back further to the initial variables:

$$\begin{cases} x+y=5, \\ xy=6. \end{cases} \Rightarrow \begin{cases} x=5-y, \\ (5-y)y=6. \end{cases} \Rightarrow \begin{cases} x=5-y, \\ y^2-5y+6=0. \end{cases} \Rightarrow \begin{cases} x_1=3, x_2=2, \\ y_1=2, y_2=3. \end{cases}$$

So, the solution of the original system is: $\{(3,2);(2,3)\}$.

5.9. Irrational systems

Example 5.16. Solve the system:

$$\begin{cases} \sqrt[4]{x+y}+\sqrt[4]{x-y}=4, \\ \sqrt{x+y}-\sqrt{x-y}=8. \end{cases}$$

Solution: Let

$$\sqrt[4]{x+y}=u, \quad \sqrt[4]{x-y}=v \quad (u\geq 0, v\geq 0).$$

So the system takes the form:

$$\begin{cases} u+v=4, \\ u^2-v^2=8, \end{cases} \Rightarrow \begin{cases} u+v=4, \\ (u+v)(u-v)=8, \end{cases} \Rightarrow \begin{cases} u+v=4, \\ u-v=2, \end{cases} \Rightarrow \begin{cases} v=1, \\ u=3. \end{cases}$$

Returning to the variables x, y, we have:

$$\begin{cases} \sqrt[4]{x+y}=3, \\ \sqrt[4]{x-y}=1. \end{cases} \Rightarrow \begin{cases} x+y=81 \\ x-y=1 \end{cases} \Rightarrow \begin{cases} x=41, \\ y=40. \end{cases}$$

So, the solution to the system is: $\{(41;40)\}$.

Example 5.17. Solve the system:

$$\begin{cases} \sqrt{\dfrac{20y}{x}}=\sqrt{x+y}+\sqrt{x-y}, \\ \sqrt{\dfrac{16x}{5y}}=\sqrt{x+y}+\sqrt{x-y}. \end{cases}$$

Solution: Multiply the left and right sides of the equations of this system:

$$\sqrt{\frac{20\cdot 16}{5}}=x+y-x+y,$$

wherefrom: $y=4$.

Substituting this value into the second equation, we get:

$$\sqrt{\frac{4x}{5}} = \sqrt{x+4} - \sqrt{x-4}.$$

There is an obvious restriction: $x \ge 4$. Let's square both sides of the equation:

$$\frac{4x}{5} = x + 4 - 2\sqrt{x^2 - 16} + x - 4,$$

Then, $10\sqrt{x^2-16} = 6x$, or $5\sqrt{x^2-16} = 3x$. Square further both sides: $25\left(x^2 - 16\right) = 9x^2$, $x = \pm 5$. Taking into account the restriction $x \ge 4$, we have: $x = 5$.

So, the solution is: $\{(5;4)\}$.

5.10. Solving problems on the compilation of equations

Example 5.18. Two workers together made 72 parts per shift. After the first worker increased labor productivity by 15 %, and the second one by 25 %, they together produced 86 parts per shift. How many parts did each worker make before the increase in productivity?

Solution: The number of parts that the first worker makes per shift is denoted by *x*, and the second one by *y*. Then together they will make: $x + y = 72$. After increasing the labor productivity by 15 %, the first worker will produce 1,15*x* parts per shift. After increasing productivity by 25 %, the second worker will produce 1,25*y* parts per shift, and together they will produce:

$$1{,}15x + 1{,}25y = 86 \text{(parts)}.$$

So, we have the following system:

$$\begin{cases} x + y = 72, \\ 1{,}15x + 1{,}25y = 86. \end{cases} \quad \begin{cases} -1{,}15x - 1{,}15y = -82{,}8, \\ 1{,}15x + 1{,}25y = 86. \end{cases} \Rightarrow \begin{cases} x = 72 - y \\ 0{,}1y = 3{,}2 - y, \end{cases}$$

$$\Rightarrow \begin{cases} x = 40, \\ y = 32. \end{cases}$$

So, the first worker made 40 parts per shift, and the second one – 32 parts.

Example 5.19. In two alloys, the ratio of copper and zinc is 5 : 2 and 3 : 4 respectively. How much of each alloy must be taken to get 28 kg of a new alloy with the same content of copper and zinc?

Solution: Let the quantity of the alloy of the first type which must be taken be denoted by x, of the second type by y.

In the first alloy, the copper and zinc content is 5 : 2; then, the amount of copper in it is $\frac{5x}{7}$, and zinc – $\frac{2x}{7}$. In the second alloy the content of copper and zinc is 3 : 4; the amount of copper in it is $\frac{3y}{7}$, and zinc – $\frac{4y}{7}$. The new alloy contains $\frac{5x}{7}+\frac{3y}{7}$ of copper and $\frac{2x}{7}+\frac{4y}{7}$ of zinc. Their amount is the same. Then, $\frac{5x}{7}+\frac{3y}{7}=\frac{2x}{7}+\frac{4y}{7}$, wherefrom, $3x=y$.

Make a system of equations:

$$\begin{cases} 3x=y, \\ x+y=28, \end{cases} \Rightarrow \begin{cases} 3x=y, \\ x+3x=28, \end{cases} \Rightarrow \begin{cases} y=21, \\ x=7. \end{cases}$$

So, 7 kg of the first alloy and 21 kg of the second one must be taken.

Example 5.20. A train travels at a constant speed. If it increased the speed by 6 km/h, it would cover this way 4 hours faster, and if it reduced the speed by 6 km/h, it would be on route 6 hours longer. Find the length of the path.

Solution: The constant speed at which the train moves is denoted by v, and the path by S. According to the condition of the problem, compose a system of equations:

$$\begin{cases} \frac{S}{v}-\frac{S}{v+6}=4, \\ \frac{S}{v-6}-\frac{S}{v}=6, \end{cases} \Rightarrow \begin{cases} S\left(\frac{1}{v}-\frac{1}{v+6}\right)=4, \\ S\left(\frac{1}{v-6}-\frac{1}{v}\right)=6. \end{cases}$$

Dividing one equation by another, we have:

$$\frac{\frac{1}{v}-\frac{1}{v+6}}{\frac{1}{v-6}-\frac{1}{v}}=\frac{4}{6} \quad \Rightarrow \quad \frac{(v+6-v)(v-6)v}{v(v+6)(v-v+6)}=\frac{2}{3} \quad \Rightarrow \quad \frac{(v-6)}{(v+6)}=\frac{2}{3} \Rightarrow$$

$$\Rightarrow 3v-18=2v+12,$$

wherefrom: $v=30$ km/h.

From the first equation of the system, we find S:

$$S\left(\frac{1}{30}-\frac{1}{36}\right)=4 \quad \Rightarrow \quad S=720\,\text{km}.$$

So, the path length is 720 km.

Tasks for individual work

5.21. For what value of a does the system

$$\begin{cases} ax+9y=5, \\ 4x+ay=1 \end{cases}$$

have a unique solution?

5.22. How many solutions does the system

$$\begin{cases} 3x+4y=6, \\ 6x+8y=12 \end{cases}$$

have?

Solve the systems

5.23.

$$\begin{cases} x+y=-1 \\ 2x-5y=19. \end{cases}$$

5.24.

$$\begin{cases} 3x+4y=16 \\ 2x-15y=-7. \end{cases}$$

5.25.

$$\begin{cases} x+y=3 \\ |x|+2y=4. \end{cases}$$

5.26.

$$\begin{cases} 2x-3y=-5 \\ 4x+3|y|=17. \end{cases}$$

5.27.

$$\begin{cases} |x-1|+y=0 \\ 2x-y=1. \end{cases}$$

5.28.

$$\begin{cases} |y|+x=3 \\ y-2|x|=-3. \end{cases}$$

5.29.

$$\begin{cases} x+2y=6 \\ |x-3|-y=0. \end{cases}$$

5.30.

$$\begin{cases} |x|+2|y|=3 \\ 7x+5y=2. \end{cases}$$

For what value of a do the systems have a unique solution? Find these solutions.

5.31.

$$\begin{cases} x+ay=1, \\ ax+y=2a. \end{cases}$$

5.32.

$$\begin{cases} (a+1)x-y=a+1, \\ x+(a-1)y=2. \end{cases}$$

Solve the following systems:

5.33.

$$\begin{cases} \dfrac{15}{x-y}-\dfrac{4}{3x+y}=1, \\ \dfrac{45}{x-y}-\dfrac{7}{3x+y}=8. \end{cases}$$

5.34.

$$\begin{cases} \dfrac{4}{2x+y+2}-\dfrac{5}{2x-y+5}=-3, \\ \dfrac{6}{2x+y+2}+\dfrac{2}{2x-y+5}=5. \end{cases}$$

Solve the systems using the Cramer's formula:

5.35.

$$\begin{cases} 2x-y=12, \\ 3x+2y=11. \end{cases}$$

5.36.

$$\begin{cases} 4x+3y=1, \\ 5x-2y=7. \end{cases}$$

5.37.

$$\begin{cases} 8x - 7y = 10, \\ 3x + 4y = 17. \end{cases}$$

5.38.

$$\begin{cases} 5x - 4y = -1, \\ 24x - 9y = 2. \end{cases}$$

5.39.

$$\begin{cases} (a+4)x + 3y = a + 1, \\ ax + (a-1)y = a - 1. \end{cases}$$

5.40.

$$\begin{cases} (1-a)x + 2y = 0, \\ x + (2-a)y = 0. \end{cases}$$

Solve the systems using the Gauss' formula:

5.41.

$$\begin{cases} x_1 + x_2 - 3x_3 = 6, \\ 2x_1 - 2x_2 + x_3 = -3, \\ x_1 + 4x_2 - x_3 = 10. \end{cases}$$

5.42.

$$\begin{cases} x_1 - 4x_2 + 5x_3 = -5, \\ 2x_1 + 3x_2 - 3x_3 = 1, \\ 3x_1 + x_2 + x_3 = -2. \end{cases}$$

5.43.

$$\begin{cases} 2x_1 - 3x_2 + x_3 = 3, \\ x_1 + 4x_2 - 2x_3 = 4, \\ x_1 - 2x_2 + 5x = -3 \end{cases}$$

5.44.

$$\begin{cases} x_1 + 3x_2 - x_3 = 6, \\ 4x_1 - x_2 + 2x_3 = 5, \\ 3x_1 + x_2 + x_3 = 6. \end{cases}$$

Solve the systems:

5.45.

$$\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{13}{6}, \\ x + y = 5. \end{cases}$$

5.46.

$$\begin{cases} (x+1)(y+1) = 10, \\ (x+y)(xy+1) = 25. \end{cases}$$

5.47.

$$\begin{cases} \sqrt{\dfrac{x}{y}} - \sqrt{\dfrac{y}{x}} = \dfrac{3}{2}, \\ x + y + xy = 9. \end{cases}$$

5.48.

$$\begin{cases} x^2 + y^2 = 68, \\ xy = 16. \end{cases}$$

5.49.

$$\begin{cases} 2x^2 + y^2 + 3xy = 12, \\ 2(x+y)^2 - y^2 = 14. \end{cases}$$

5.50.

$$\begin{cases} 2y^2 - 4xy + 3x^2 = 17, \\ y^2 - x^2 = 16. \end{cases}$$

5.51.

$$\begin{cases} 2x^2 + y^2 + 3xy = 12, \\ 2(x+y)^2 - y^2 = 14. \end{cases}$$

5.52.

$$\begin{cases} x^2 + y^2 = 34, \\ x + y + xy = 23. \end{cases}$$

5.53.

$$\begin{cases} \sqrt{x} + \sqrt{y} = 10, \\ \sqrt[4]{x} + \sqrt[4]{y} = 4. \end{cases}$$

5.54.

$$\begin{cases} \sqrt{\dfrac{2x-1}{y+2}} + \sqrt{\dfrac{y+2}{2x-1}} = 2, \\ x + y = 2. \end{cases}$$

Tasks on the compilation of systems of equations

5.55. A worker produced a certain number of parts in a given period. If he had produced 10 parts more per day, he would have done this work 4.5 days ahead of time, and if he had produced 5 parts fewer per day, he would have been 3 days late of the deadline. How many parts did the worker produce and in what time?

5.56. Two tractors of different capacities, working together, can plow a field in 4 days. If one tractor plowed 2/3 of the field and then the second one did the rest, the whole field would be plowed in 8 days. In how many days can each tractor plow a field?

5.57. Two workers did some work. When the first worked 2 hours, and the second 5 hours, they completed half of the work. After they had worked together for 3 hours, they had to do 1/20 of all the work. In how many hours can each worker do all the work?

5.58. The road from A to B has a length of 11.5 km. It first goes up, then on a flat terrain, and then down. A pedestrian walked a distance from A to B 2 hours 54 minutes, back from B to A – 3 hours 6 minutes. The pedestrian speed is 3 km/h up the road, 4 km/h on a flat ground, 5 km/h down the road. How long is the road on the flat ground?

5.59. In 5 hours a motorcyclist rides 259 km more than a cyclist in 4 hours. In 10 hours a cyclist rides 56 km more than a motorcyclist in 2 hours. Determine the speed of the cyclist and the motorcyclist.

5.60. The price of 60 copies of the first volume and 75 copies of the second volume is 4050 UAH. However, at a 15 % discount on the first volume and a 10 % discount on the second volume, you need to pay 3555 UAH. Determine the price of each volume.

Questions for self-assessment

1. What are the systems of linear equations?
2. Systems of nonlinear equations resulting in linear equations.
3. What is a determinant?
4. What is the Cramer's formula?
5. What is the Gauss' method?
6. What are the nonlinear systems?
7. What are the homogeneous systems of equations?
8. What are the symmetric systems of equations?
9. What are the irrational systems?

6. Algebraic inequalities

6.1. Inequality. Basic concepts

A symbolic notation in which two numbers or two expressions are joined by the sign ">" or "<" is called inequality.

Inequalities connected by "≥" or "≤" are called weak, and inequalities connected by ">" or "<" are called strict.

An inequality of the form $x < a < y$ is called a double inequality.

Inequalities containing only numbers are called numeric. Moreover, an inequality is called true if it is a true statement.

The inequality containing literal expressions is true only for certain values of the letters included in it. For example, the inequality $(a+b)^2 \geq 0$ is true for any values of a and b; the inequality $x^2 > 0$ is true for any values of x, except for zero.

6.2. The properties of numerical inequalities

1. If $a > b$, then $b < a$.
2. If $a > b$ and $b > c$, then $a > c$.
3. If the same number it added to both sides of the inequality, the sign of the inequality will not change, i.e. if $a > b$, then $a + c > b + c$.

4. If both sides of the inequality are multiplied or divided by the same positive number, then we get the inequality of the same meaning: if $a > b$ and $m > 0$, then $am > bm$ and $\frac{a}{m} > \frac{b}{m}$.

5. If both sides of the inequality are multiplied or divided by the same negative number, then the sign of the resulting inequality needs to be changed to the opposite: if $a > b$ and $m < 0$, then $am < bm$ and $\frac{a}{m} < \frac{b}{m}$.

6. Inequalities of the same meaning can be added termwise: if $a > b$ and $c > d$, then $a + c > b + d$; if $a < b$ and $c < d$, then $a + c < b + d$.

7. Inequalities of the opposite meaning can be subtracted term by term, with the sign of the inequality that from which the subtraction is made: if $a > b$ and $c < d$, then $a - c > b - d$.

8. Both parts of the inequality with positive members can be raised to the same natural power: if $a > b$ and $a > 0$, $b > 0$, then $a^n > b^n$, $n \in N$.

9. The converse is also true: if $a^n > b^n$, $a > 0$ and $b > 0, n \in N$, then $a > b$.

6.3. Linear inequalities

Further, inequalities containing a variable will be considered.

To solve such an inequality means to find the set of variable values for which this inequality is true. The elements of this set are called solutions of inequality. Inequalities are called equivalent if the sets of their solutions coincide.

The solution of inequalities is based on their properties (see 6.2).

In general, a linear inequality has the form: $ax + b > 0$ (or $ax + b < 0$), where a and b are real numbers.

If $a > 0$, then $x > -\frac{b}{a}$; if $a < 0$, then $x < -\frac{b}{a}$.

Example 6.1. Solve the inequalities:

a) $10(x-2)+5 > 2x+5(x+1)$; b) $\frac{x-1}{2} - \frac{x+2}{3} \le \frac{x-3}{4} + x$.

Solution: Let us transform each of the inequalities to the form $ax+b>0$ or $ax+b\le 0$.

a) $10x-20+5>2x+5x+5$, $3x>20$, $x>\frac{20}{3}$ or $x\in\left(\frac{20}{3};\infty\right)$.

b) $6(x-1)-4(x+2)\le 3(x-3)+12x$;

$6x-6-4x-8\le 3x-9+12x$; $-13x\le 5$; $x\ge -\frac{5}{13}$, i.e. $x\in[-\frac{5}{13};\infty)$.

Example. 6.2. Solve and investigate the inequalities:

a) $m+x<2-mx$; b) $2(x-a)>ax-4$.

Solution: a) Let us transform the inequality:

$x+mx<2-m$; $(m+1)x<2-m$.

If $m+1>0$, i.e. $m>-1$, then $x<\frac{2-m}{m+1}$.

If $m+1<0$, i.e. $m<-1$, then $x>\frac{2-m}{m+1}$.

If $m+1=0$, i.e. $m=-1$, then $0x<2+1$, which is true for any $x\in R$.

b) Let us write down:

$2x-2a>ax-4 \Leftrightarrow 2x-xa>2a-4 \Leftrightarrow (2-a)x>2a-4 \Leftrightarrow (2-a)x>2(a-2)$.

If $2-a>0$, i.e. $a<2$, then $x>\frac{2(a-2)}{2-a}$, $x>-2$.

If $2-a<0$, i.e. $a>2$, then $x<-2$.

If $2-a=0$, i.e. $a=2$, then $0\cdot x>0$, which is not true. It means that there are no solutions for $a=2$.

6.4. Linear inequality systems

Several inequalities with one variable form a system if you need to find common solutions to given inequalities. The value of a variable in which each of the inequalities turns into a valid numerical inequality is called a solution to the system of inequalities. The solution to a system of inequalities is the intersection of the solution sets for each inequality. Inequalities included in the system are combined by curly braces or signs of double inequality. For example,

$$\begin{cases} 6x+2>3x-4, \\ 2x+1>4x-7 \end{cases} \Leftrightarrow \begin{cases} x>-2 \\ x<4 \end{cases} \Leftrightarrow -2<x<4.$$

Example 6.3. Solve the inequality systems:

a) $\begin{cases} 2x>3x+4, \\ 5x+3\geq 8x+21; \end{cases}$ b) $\begin{cases} 10(x-1)+11>4x+5(x+1), \\ 3x-5<2(x-1); \end{cases}$

c) $\begin{cases} 2x+3>18-3x, \\ 5x-2\geq 2x+1; \end{cases}$ d) $\begin{cases} 3x+7\geq 7x-9, \\ x-3>-3x+1. \end{cases}$

Solution: After transformations the systems take the form:

a) $\begin{cases} -x>4, \\ -3x\geq 18 \end{cases} \Leftrightarrow \begin{cases} x<-4, \\ x\leq -6, \end{cases} \quad x\leq -6, \quad x\in(-\infty;-6].$

b) $\begin{cases} x>6, \\ x<3; \end{cases}$ there are no solutions.

c) $\begin{cases} 5x>15, \\ 3x\geq 3; \end{cases} \begin{cases} x>3, \\ x\geq 1, \end{cases} \Leftrightarrow \quad x\in(3;\infty).$

d) $\begin{cases} 4x\leq 16, \\ 4x>4; \end{cases} \begin{cases} x\leq 4, \\ x>1, \end{cases} \Leftrightarrow \quad x\in(1;4].$

Example 6.4. Find the domains of the functions:

a) $y=\sqrt{3x-1}-2\sqrt[6]{5-2x};$ b) $y=\dfrac{\sqrt{2x+1}}{\sqrt[4]{2-3x}}.$

Solution: The domain of each function can be found by solving the system of inequalities:

a) $\begin{cases} 3x-1\geq 0, \\ 5-2x\geq 0; \end{cases} \Leftrightarrow \begin{cases} x\geq \frac{1}{3} \\ x\leq \frac{5}{2} \end{cases} \Leftrightarrow \frac{1}{3}\leq x\leq \frac{5}{2},\ D(y)=\left[\frac{1}{3};\frac{5}{2}\right].$

b) $\begin{cases} 2x+1\geq 0, \\ 2-3x>0; \end{cases} \Leftrightarrow \begin{cases} x\geq -\frac{1}{2}, \\ x<\frac{2}{3}. \end{cases}$

The domain of the function b) is shown in Fig. 6.1.

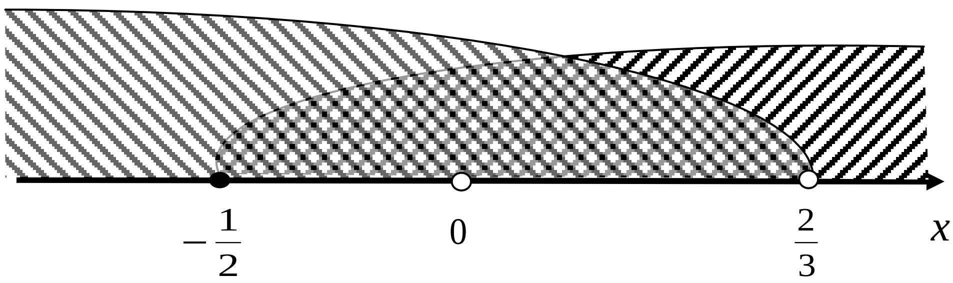


Fig. 6.1. **The domain of the function b)**

As shown in Fig. 6.1, the domain is: $D(y)=[-\frac{1}{2};0)\cup\left(0;\frac{2}{3}\right)$.

If for two or more inequalities you need to find a set of variable values, each of which is a solution to at least one of the inequalities, you need to solve a set of inequalities.

Inequalities in the aggregate are combined by a square bracket or the word "or". For example, the record $\left[\begin{array}{l} 2x-5>4, \\ 3x+1<5 \end{array}\right.$ means that these inequalities form a system. In addition, this record can be presented as follows: ($2x-5>4$ or $3x+1<5$).

A solution of a set of inequalities is the value of a variable with which at least one of the inequalities of a set turns into a true inequality. The set of solutions of a set of inequalities is the union of the sets of solutions that comprise its inequalities.

Example 6.5. Solve the systems of inequalities:

a) $\left[\begin{array}{l} 2-3x<8-5x, \\ 2(4-x)\geq x-22; \end{array}\right.$ b) $\left[\begin{array}{l} \dfrac{2x}{3}+\dfrac{x}{4}>x-1, \\ \dfrac{x+1}{2}<\dfrac{x-1}{3}+x. \end{array}\right.$

Solution: Let us transform the inequalities of each aggregate, solve them and combine the results obtained:

a) $\left[\begin{array}{l} 2x<6, \\ -3x\geq -30; \end{array}\right.$ $\left[\begin{array}{l} x<3, \\ x\leq 10; \end{array}\right.$ wherefrom: $x\leq 10$, i.e. $x\in(-\infty;10]$.

Graphically, the solution of the aggregate is the areas covered with hatching (Fig. 6.2).

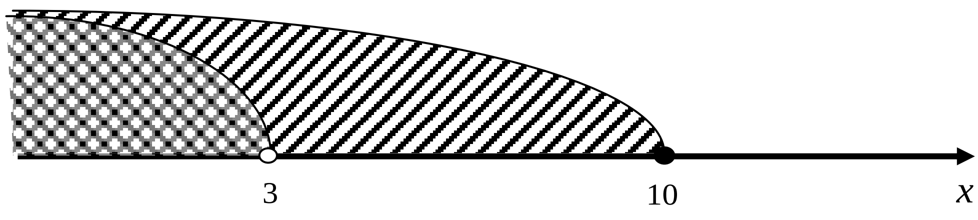


Fig. 6.2. **The solution of the system of inequalities a) by the interval method**

b) $\left[\begin{array}{l} 8x+3x>12x-12, \\ 3x+3<2x-2+6x; \end{array}\right.$ $\left[\begin{array}{l} x<12, \\ x>1. \end{array}\right.$

The solution of the system of inequalities b) is presented in Fig. 6.3.

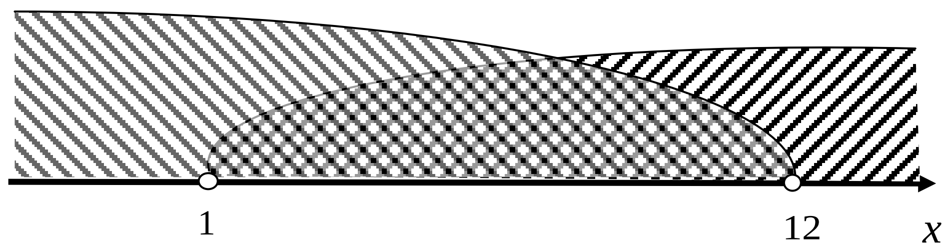


Fig. 6.3. **The solution of the system of inequalities b) by the interval method**

The following inequalities can be transformed to the system of inequalities:

1) $f(x)\cdot g(x)\geq 0 \quad \Leftrightarrow \quad \begin{cases} f(x)\geq 0, \\ g(x)\geq 0; \end{cases}$ or $\begin{cases} f(x)\leq 0, \\ g(x)\leq 0; \end{cases}$

2) $f(x)\cdot g(x)\leq 0 \quad \Leftrightarrow \quad \begin{cases} f(x)\geq 0, \\ g(x)\leq 0; \end{cases}$ or $\begin{cases} f(x)\leq 0, \\ g(x)\geq 0; \end{cases}$

3) $\dfrac{f(x)}{g(x)}\geq 0 \quad \Leftrightarrow \quad \begin{cases} f(x)\geq 0, \\ g(x)> 0; \end{cases}$ or $\begin{cases} f(x)\leq 0, \\ g(x)< 0; \end{cases}$

4) $\dfrac{f(x)}{g(x)}\leq 0 \quad \Leftrightarrow \quad \begin{cases} f(x)\geq 0, \\ g(x)< 0; \end{cases}$ or $\begin{cases} f(x)\leq 0, \\ g(x)> 0; \end{cases}$

Note that $f(x)$ and $g(x)$ are not necessarily linear functions.

Example 6.6. Solve the inequalities:

a) $\dfrac{4-x}{1-3x}>0$; b) $\dfrac{2x+3}{3x+2}\geq 2$.

Solution: a) Study the system of inequalities:

$$\frac{4-x}{1-3x}>0 \quad \Leftrightarrow \quad \frac{x-4}{3x-1}>0 \quad \Leftrightarrow \quad \begin{cases} x-4>0, \\ 3x-1>0; \end{cases} \text{ or } \begin{cases} x-4<0, \\ 3x-1<0; \end{cases}$$

$$\begin{cases} x>4, \\ x>\frac{1}{3}; \end{cases} \text{ or } \begin{cases} x<4, \\ x<\frac{1}{3}; \end{cases} \Leftrightarrow \quad x>4 \text{ or } x<\frac{1}{3} \quad \Leftrightarrow \quad \left[\begin{array}{l} x>4, \\ x<\frac{1}{3}; \end{array}\right. \text{ i.e.}$$

$$x\in\left(-\infty;\frac{1}{3}\right)\cup(4;\infty).$$

b) $\frac{2x+3}{3x+2}\geq 2 \Leftrightarrow \frac{2x+3}{3x+2}-2\geq 0 \Leftrightarrow \frac{-4x-1}{3x+2}\geq 0 \Leftrightarrow \frac{4x+1}{3x+2}\leq 0$.

Let us apply rule 4) to the last inequality:

$$\frac{4x+1}{3x+2}\leq 0 \quad \Leftrightarrow \quad \begin{cases} 4x+1\geq 0, \\ 3x+2<0; \end{cases} \text{ or: } \begin{cases} 4x+1\leq 0, \\ 3x+2>0; \end{cases} \Leftrightarrow \quad \begin{cases} x\geq -\frac{1}{4}, \\ x<-\frac{2}{3}; \end{cases} \text{ or:}$$

$$\begin{cases} x\leq -\frac{1}{4}, \\ x>-\frac{2}{3}; \end{cases} \Leftrightarrow \quad x\in \text{Ø} \text{ or: } -\frac{2}{3}<x\leq -\frac{1}{4}, \text{ i.e., } \quad x\in(-\frac{2}{3};-\frac{1}{4}].$$

6.5. Inequalities containing a variable under the sign of the module

Recall the definition of the module:

$$|f(x)|=\begin{cases} f(x), \ if \ f(x)\geq 0, \\ -f(x), if \ f(x)<0. \end{cases}$$

When solving inequalities with a module, it is advisable to use the following equivalent inequalities:

$$|f(x)|<g(x)\Leftrightarrow\begin{cases} f(x)<g(x), \\ f(x)>-g(x); \end{cases} \tag{6.1}$$

$$|f(x)|>g(x)\Leftrightarrow\left[\begin{array}{l} f(x)>g(x), \\ f(x)<-g(x); \end{array}\right. \tag{6.2}$$

Example 6.7. Solve the inequality:

$$|3x+4| \le x+5.$$

Solution: Let us proceed to the system of inequalities:

$$|3x+4| \le x+5 \;\Leftrightarrow\; \begin{cases} 3x+4 \le x+5 \\ 3x+4 \ge -(x+5) \end{cases} \;\Leftrightarrow\; \begin{cases} x \le \dfrac{1}{2}, \\ x \ge -\dfrac{9}{4}; \end{cases} \Rightarrow x \in \left[-\frac{9}{4}; \frac{1}{2}\right].$$

Example 6.8. Solve the inequality:

$$|x| > 3(x+2)+7.$$

Solution: Let us transform the inequality and proceed to the system of inequalities of the form (6.2):

$$|x| > 3x+13 \;\Leftrightarrow\; \left[\begin{array}{l} x > 3x+13, \\ x < -3x-13; \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x < -\dfrac{13}{2}, \\ x < -\dfrac{13}{4}; \end{array}\right. \Leftrightarrow\; x < -\frac{13}{4}, \text{i.e.}$$

$$x \in \left(-\infty; -\frac{13}{4}\right).$$

Example 6.9. Solve the inequality:

$$|1-3x| - |x+2| \le 2.$$

Solution: We apply the interval method to solve this inequality. On the numerical axis, mark the points at which each of the modules vanishes: $x=-2$ and $x=\frac{1}{3}$. With these points, the numerical axis will be divided into three intervals (Fig. 6.4).

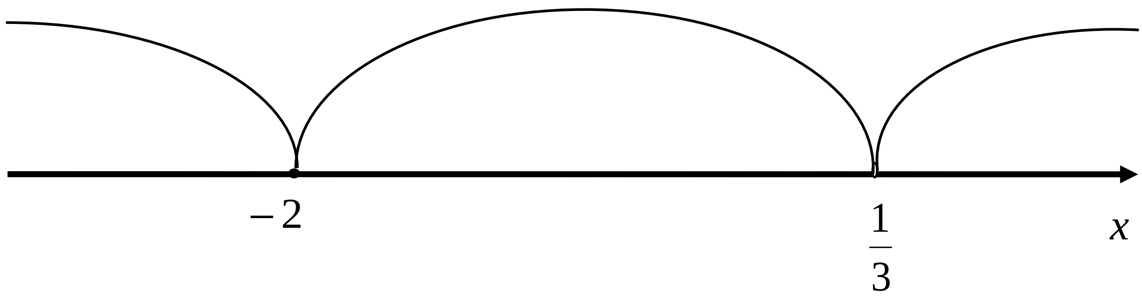


Fig. 6.4. **The solution of the inequality by the interval method for** $x=-2$**,** $x=\frac{1}{3}$

Let us solve this inequality in each of these intervals. Let $x \le -2$. Then:

$$\begin{cases} x \le -2, \\ 1-3x+x+2 \le 2; \end{cases} \Leftrightarrow \begin{cases} x \le -2, \\ x \ge \frac{1}{2}, \end{cases} \quad x \in \varnothing.$$

Let $-2 < x \le \frac{1}{3}$. Then,

$$\begin{cases} -2 < x \le \frac{1}{3}, \\ 1-3x-x-2 \le 2; \end{cases} \Leftrightarrow \begin{cases} x \le \frac{1}{3}, \\ x \ge -\frac{3}{4}, \end{cases} \quad x \in \left[-\frac{3}{4}; \frac{1}{3}\right]$$

Fig. 6.5 shows the solution of the inequality by the interval method:

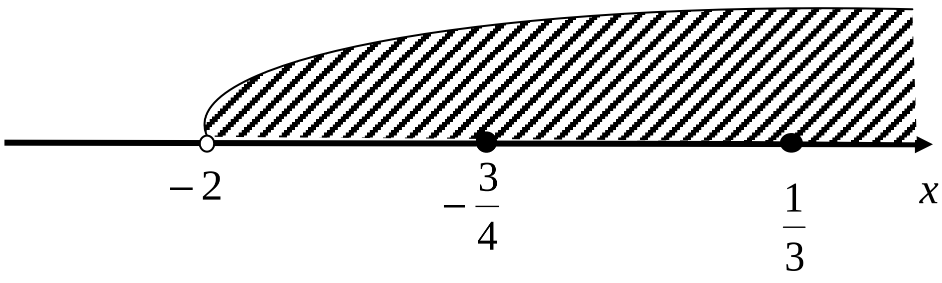


Fig. 6.5. **The solution of the inequality by the interval method**

Let $x > \frac{1}{3}$. Then,

$$\begin{cases} x > \frac{1}{3}; \\ 3x - 1 - x - 2 \le 2; \end{cases} \Leftrightarrow \begin{cases} x > \frac{1}{3}; \\ x \le \frac{5}{2}; \end{cases} \Leftrightarrow x \in (\frac{1}{3}; \frac{5}{2}].$$

Combine the results:

$$x \in \left[-\frac{3}{4}; \frac{1}{3}\right] \cup (\frac{1}{3}; \frac{5}{2}] = \left[-\frac{3}{4}; \frac{5}{2}\right].$$

Tasks for individual work

6.10. Solve the inequalities:

1) $6 \cdot (2x + 7) < 15 \cdot (x + 2);$ 2) $5 \cdot (x + 3) - \frac{x - 7}{8} > \frac{11}{2} \cdot (x - 2);$

3) $\frac{5}{3}x + \frac{7 - 3x}{2} \le x + \frac{x}{6} - \frac{3x + 7}{3};$ 4) $(x - 3)(x - 5) > (x + 1)(x - 4) - 5x.$

6.11. Solve and investigate the inequalities:

1) $3(2a - x) < ax + 1;$ 2) $a + x < 2 - ax$.

6.12. Solve the systems:

1) $\begin{cases} x - 4 < 8 \\ 2x - 5 < 13, \\ 3 - x > 1, \end{cases}$ 2) $\begin{cases} 2x - 1 < x + 3 \\ 5x - 1 < 6 - 2x, \\ x - 3 < 0, \end{cases}$ 3) $\begin{cases} 3x + 2 > x - 2, \\ x + 15 > 6 - 2x, \\ x - 14 < 5x + 14, \end{cases}$

4) $\begin{cases} 5(x + 1) - 9x - 3 > -6(x + 2); \\ 3(3 + 2x) < 7x - 2(x - 8), \end{cases}$ 5) $\begin{cases} 5x - 3 > 1 + x \\ \frac{1}{2} - 3x < \frac{2}{3}x - 5, \end{cases}$

6) $\begin{cases} \dfrac{4x-3}{6}+3>\dfrac{3x}{2}+\dfrac{5}{8}, \\ \dfrac{4x-3}{8}+\dfrac{x-5}{5}>\dfrac{x-1}{2}. \end{cases}$

6.13. Solve the systems:

1) $\left[\begin{array}{l} 4x+7>2x+13, \\ 3x+2<2x+3; \end{array}\right.$ 2) $\left[\begin{array}{l} 5-2x>3x-10; \\ 2(1-2x)>1-5x, \end{array}\right.$

3) $\left[\begin{array}{l} 2(x-2)>5-x; \\ 1-5x<4(2-x), \end{array}\right.$ 4) $\left[\begin{array}{l} \dfrac{2x}{3}+\dfrac{x}{4}>x-1, \\ \dfrac{x+1}{2}<\dfrac{x-1}{3}+x. \end{array}\right.$

6.14. Solve the systems:

1) $\dfrac{2-x}{x-4}\geq 0;$ 2) $\dfrac{1-2x}{4-2x}\leq 0;$ 3) $\dfrac{3x+1}{2x-5}>2;$ 4) $\dfrac{x+4}{2x+3}\leq -2$.

6.15. Solve the systems:

1) $|2x+7|<5;$ 2) $|2x-2|\geq 1;$ 3) $|2x-4|<x-1;$
4) $2|x+1|>x+4;$ 5) $|x+2|-|x-1|<x-1{,}5;$ 6) $|x+1|+|x-4|>7$.

6.16. Find the domains:

1) $y=\sqrt{x-1}\cdot\sqrt{x+1}$; 2) $y=\sqrt{x-1}+\sqrt{6-x}$; 3) $y=\sqrt{\dfrac{x+3}{5-x}}$;

4) $y=\sqrt{2-x}-\sqrt{x+1}$; 5) $y=\sqrt{2+x}+\dfrac{1}{x-1}$.

6.6. Quadratic inequalities

A square inequality is the inequality of the form:

$$ax^2+bx+c>0 \quad \text{or} \quad ax^2+bx+c<0, a\neq 0.$$

Let us consider the graphical method for solving these inequalities for $a > 0$. If $a < 0$, then by multiplying both sides of the inequality by (−1) and changing its sign to the opposite one can obtain $a > 0$.

Let us denote: $y = ax^2 + bx + c, a > 0$. The graph of this function is a parabola. Depending on the sign of the discriminant $D = b^2 - 4ac$, consider three possible cases of the location of a parabola for $a > 0$, which is demonstrated in Fig. 6.6 – 6.8.

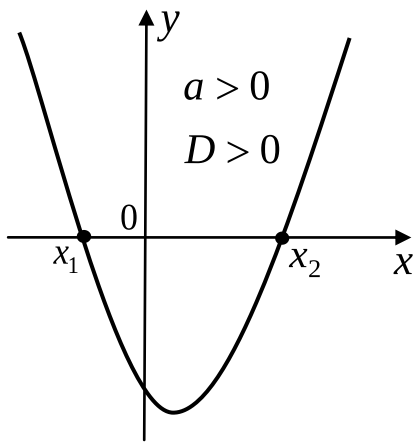


Fig. 6.6. **A square inequality for** $a > 0, D > 0$

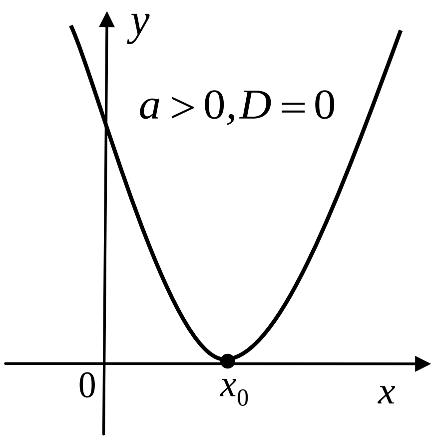


Fig. 6.7. **A square inequality for** $a > 0, D = 0$

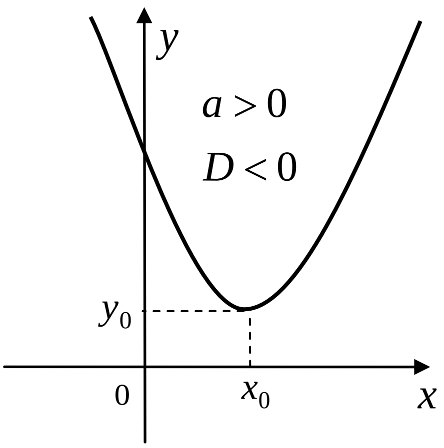


Fig. 6.8. **A square inequality for** $a > 0, D < 0$

We have to solve the inequalities: $y > 0$ and $y < 0$ for $a > 0$. From the graphs it follows that:

1) for $D > 0$ (Fig. 6.6) the inequality $ax^2 + bx + c > 0$ is true if $x \in (-\infty; x_1) \cup (x_2; \infty)$; the inequality $ax^2 + bx + c < 0$ is true if $x \in (x_1; x_2)$;

2) for $D = 0$ (Fig. 6.7) the inequality $ax^2 + bx + c > 0$ is true for $x \in R$, except $x = x_0$; the inequality $ax^2 + bx + c < 0$ has no solutions, i.e. $x \in \varnothing$;

3) for $D < 0$ (Fig. 6.8) the inequality $ax^2 + bx + c > 0$ is true for all $x \in R$; the inequality $ax^2 + bx + c < 0$ has no solutions, i.e. $x \in \varnothing$.

Note that the square trinomial $y = ax^2 + bx + c$, $a > 0$ is positive if its discriminant $b^2 - 4ac < 0$.

Example 6.17. Solve the inequalities:

1) $x^2 - 8x + 15 \geq 0$; 2) $7 - 6x - x^2 \geq 0$;

3) $x^2 + x + 1 > 0$; 4) $-4x^2 + 4x - 1 < 0$.

Solutions:

1) Let us solve the equation $x^2 - 8x + 15 = 0$. The solution: $x_1 = 3,\ x_2 = 5$. On the number axis Ox, we construct a parabola. (Fig. 6.9) (there is no need to draw the axis Oy, because the top of the parabola is not involved in the solution).

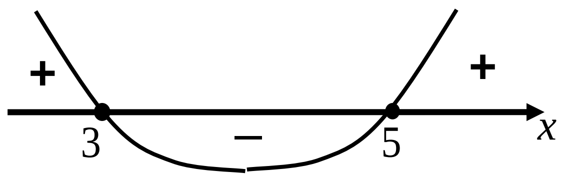


Fig. 6.9. **A square inequality for** $x_1 = 3,\ x_2 = 5$

The figure shows that the solution of the inequality is $x \in (-\infty; 3] \cup [5; \infty)$. Note that drawing is not necessary – you can use Fig. 6.6 – 6.8.

2) Let us transform the inequality $7 - 6x - x^2 \geq 0$ to the form $x^2 + 6x - 7 \leq 0$ and find the roots of the equation $x^2 + 6x - 7 = 0$; these are: $x_1 = -7$ and $x_2 = 1$ (Fig. 6.10).

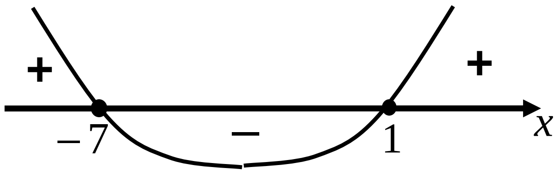


Fig. 6.10. **A square inequality for** $x_1 = -7$**,** $x_2 = 1$

As we can see, the solution is: $x \in [-7;1]$.

3) The inequality $x^2 + x + 1 > 0$ is true for all $x \in R$, because its discriminant is negative: $D = b^2 - 4ac = 1 - 4 = -3 < 0$.

4) Let us transform the inequality to the form: $4x^2 - 4x + 1 > 0$. Its discriminant is zero: $D = b^2 - 4ac = 16 - 16 = 0$. This means that the square trinomial is positive: $4x^2 - 4x + 1 > 0$ for all $x \in R$ except for $x_0 = \frac{1}{2}$, where x_0 is the root of the equation $4x^2 - 4x + 1 = 0$. So, the solution is:

$$x \in \left(-\infty; \frac{1}{2}\right) \cup \left(\frac{1}{2}; \infty\right).$$

Square inequalities with $D > 0$ can be solved by the method of intervals, which is discussed in section 6.7.

Example. 6.18. Find, for which values of a the inequality

$$(a-3)x^2 - 2ax + 3a - 6 > 0$$

holds for all real values of x.

Solution: Using the above inequality and Fig. 6.8, we write the system of inequalities:

$$\begin{cases} a - 3 > 0, \\ D < 0, \end{cases} \quad \text{i.e.} \quad \begin{cases} a - 3 > 0, \\ 4a^2 - 4(a-3)(3a-6) < 0; \end{cases}$$

$$\begin{cases} a > 3, \\ -2a^2 + 15a - 18 < 0; \end{cases} \quad \begin{cases} a > 3, \\ 2a^2 - 15a + 18 > 0. \end{cases}$$

To solve the quadratic inequality, we solve the equation:

$$2a^2 - 15a + 18 = 0,$$

wherefrom: $a_1 = \frac{3}{2}, a_2 = 6$.

Fig. 6.11 shows the solution of the resulting system of inequalities.

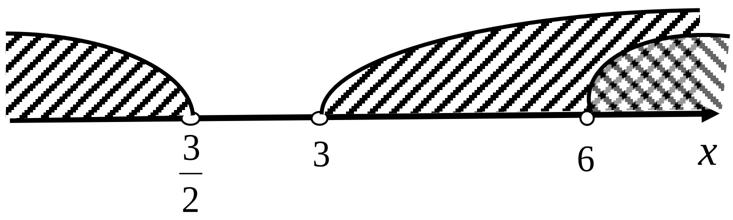


Fig. 6.11. **A square inequality for** $a_1 = \frac{3}{2}, a_2 = 6$

Then: $x \in (6;\infty)$.

Next, let's consider the examples of solving square inequalities containing an unknown under the modulus sign. In this case, we use both the definition of the module and inequalities of the form (6.1) and (6.2):

$$|f(x)| \le g(x) \Leftrightarrow \begin{cases} f(x) \le g(x), \\ f(x) \ge -g(x); \end{cases}$$

$$|f(x)| \ge g(x) \Leftrightarrow \left[\begin{array}{l} f(x) \ge g(x), \\ f(x) \le -g(x). \end{array} \right.$$

Example 6.19. Solve the equation:

$$x^2 - |x| - 2 \ge 0.$$

Solution: This inequality is equivalent to the set of the systems of inequalities:

1) $\begin{cases} x \ge 0, \\ x^2 - x - 2 \ge 0; \end{cases}$ or 2) $\begin{cases} x < 0, \\ x^2 + x - 2 \ge 0. \end{cases}$

The roots of the equation $x^2 - x - 2 = 0$ are: $x_1 = -1, x_2 = 2$. The roots of the equation $x^2 + x - 2 = 0$ are: $x_1 = -2, x_2 = 1$.

The solution of the system 1) is shown in Fig. 6.12; the solution of the system 2) is shown in Fig. 6.13.

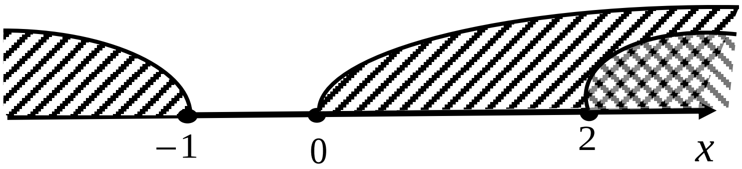


Fig. 6.12. **A square inequality for $x \geq 0$**

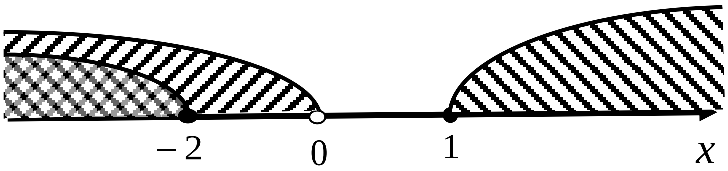


Fig. 6.13. **A square inequality for $x < 0$**

The combined solution is shown in Fig. 6.14.

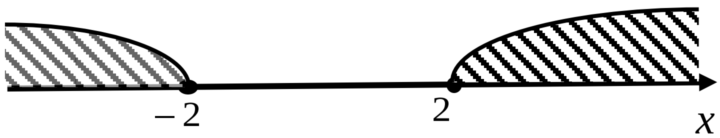


Fig. 6.14. **A square inequality (the combined solution)**

Example 6.20. Solve the inequality:

$$|x^2 - 2x - 3| < 3x - 3.$$

Solution: According to inequality (6.1), this inequality is equivalent to the system of inequalities:

$$\begin{cases} x^2 - 2x - 3 < 3x - 3, \\ x^2 - 2x - 3 > -3x + 3; \end{cases} \qquad \begin{cases} x^2 - 5x < 0, \\ x^2 + x - 6 > 0. \end{cases}$$

The roots of the equation $x^2-5x=0$ are: $x_1=0, x_2=5$. The roots of the equation $x^2+x-6=0$ are: $x_1=-3, x_2=2$.

The solution of the first inequality is $(0;5)$; the solution of the second inequality is $(-\infty;-3)\cup(2;\infty)$.

The intersection of these solutions is shown in Fig. 6.15:

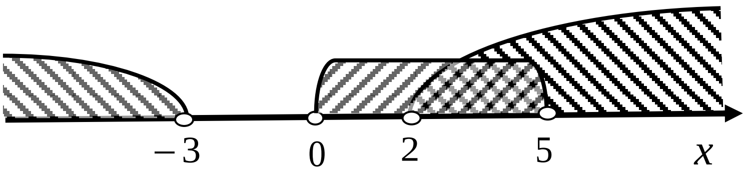


Fig. 6.15. **A square inequality** $\left|x^2-2x-3\right|<3x-3$

The solution is: $x\in(2;5)$.

Example 6.21. Solve the inequality: $\left|x^2-2x\right|>x$.

Solution: According to (6.2), this inequality is equivalent to the set of inequalities:

$$\left[\begin{array}{l} x^2-2x>x, \\ x^2-2x<-x; \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x^2-3x>0, \\ x^2-x<0; \end{array}\right. \Leftrightarrow \left[\begin{array}{l} (x-3)x>0, \\ (x-1)x<0; \end{array}\right.$$

The solution of the first inequality is $(-\infty;0)\cup(3;\infty)$. The solution of the second inequality is $(0;1)$. The combined solution is shown in Fig. 6.16.

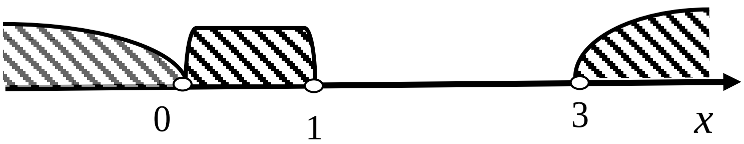


Fig. 6.16. **A square inequality** $\left|x^2-2x\right|>x$

The common solution. $(-\infty;0)\cup(0;1)\cup(3;\infty)$.

Example 6.22. Find the domain:

$$y=\sqrt{6+7x-3x^2}.$$

Solution: The task is reduced to the solution of the inequality:

$$6+7x-3x^2 \geq 0 \quad \text{or} \quad 3x^2-7x-6 \leq 0.$$

We solve the equation: $3x^2-7x-6=0$.

$$D=b^2-4ac=49+4\cdot 3\cdot 6=121 \text{(the discriminant)},$$

$$x_{1,2}=\frac{7\pm\sqrt{121}}{6}=\frac{7+11}{6}; x_1=3, x_2=-\frac{2}{3} \text{ (the solution).}$$

It is obvious that $x \in \left[-\frac{2}{3};3\right]$.

Tasks for individual work

6.23. Solve the inequalities:

1) $3x^2-7x+4 \leq 0$, 2) $8+7x-x^2 \leq 0$, 3) $x^2-3x+5>0$,
4) $x^2-14x-15>0$, 5) $x^2-x+2<0$, 6) $x^2-6x+9 \geq 0$,
7) $x^2-4x>0$, 8) $x^2<4$.

6.24. Solve the inequalities:

1) $x^2-5|x|+6<0$, 2) $|x^2-4x|<5$, 3) $|x^2-3x|<2-x$,
4) $|x^2-5x|<6$, 5) $x^2-3|x-1|>1$.

6.25. Find for which values of a the inequality $(a-1)x^2-(a+1)x+$ $+a+1>0$ holds for all real values of x.

6.7. Rational inequalities. The spacing method

The solution of inequalities of the form $f(x) \geq 0$ or $f(x) \leq 0$ is based on the important property of a continuous function: if the function $f(x)$ does not vanish on a certain interval, then it retains the sign on this interval.

The interval method is based on this property. Its essence is the following: all points on which the function vanishes or does not exist are marked on the numerical axis.

With these points, the numerical axis is divided into a number of intervals, inside each of them $f(x)$ retains the sign. To find this sign, it is sufficient to find the sign of the function at any point inside the interval under consideration. The solution of the original inequality is the union of those intervals where the sign of the function coincides with the sign of the original inequality.

Example 6.26. Solve the inequalities:

1) $3x^2 - 2x - 1 \geq 0$, 2) $\frac{3+2x}{x-4} \geq 0$, 3) $(x+2)(x-1)(x-3) > 0$,

4) $(x+1)^4(x-2) < 0$, 5) $(x+2)^2(x+1)^3(x-1)(x^2+3) \leq 0$.

Solution:

1) We find the roots of the square trinomial and decompose it into factors to make it easier to determine the sign of $f(x) = 3x^2 - 2x - 1$ at the obtained intervals: $3x^2 - 2x - 1 = 0$, $x_1 = 1, x_2 = -\frac{1}{3}$, then

$$f(x) = 3x^2 - 2x - 1 = 3(x-1)\left(x + \frac{1}{3}\right).$$

We solve further the inequality: $3(x-1)\left(x + \frac{1}{3}\right) \geq 0$. We form three intervals and put down a sign of $f(x)$ in each of them (Fig. 6.17).

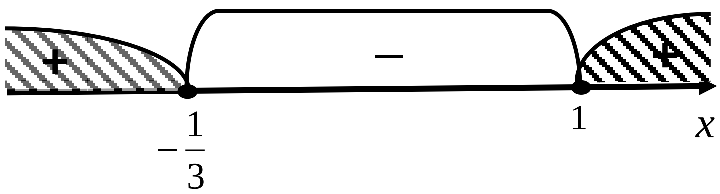


Fig. 6.17. **The spacing method for** $3x^2 - 2x - 1 \geq 0$

As shown in the graph, the solution is: $x \in (-\infty; -\frac{1}{3}] \cup [1; \infty[$.

2) Solve the inequality: $\frac{3+2x}{x-4} \geq 0$. The numerator vanishes at $x = -\frac{3}{2}$, and the denominator – at $x = 4$. We plot these points on the number axis. The points at which the denominator vanishes are marked as "o". We form three intervals and indicate the sign of the fraction in each of them (Fig. 6.18).

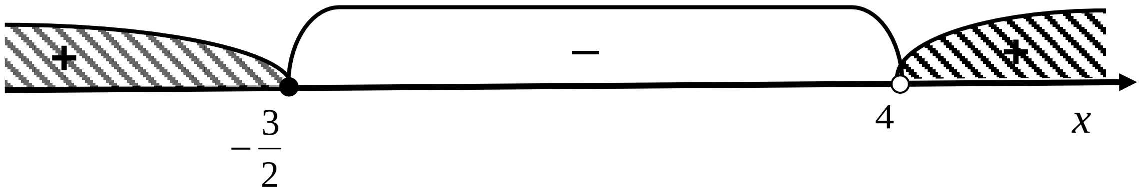


Fig. 6.18. **The spacing method for** $\frac{3+2x}{x-4}\ge 0$

We combine further the intervals where the sign "+" stands. The solution is:

$$x\in(-\infty;-\frac{3}{2}]\cup(4;\infty).$$

3) Let $f(x)=(x+2)(x-1)(x-3)$. The function $f(x)$ vanishes at $x=-2, x=1, x=3$. These points are plotted on the number axis. At the same time, the scale does not have to be observed, it is better to place the points at equal distances from each other (Fig. 6.19). All points are "punctured" because the strict inequality is solved, that is, $f(x)>0$.

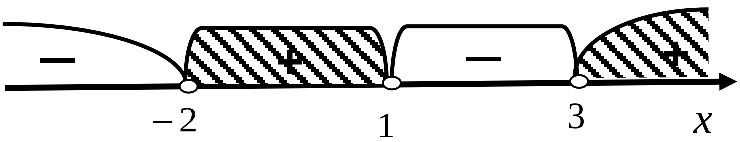


Fig. 6.19. **The spacing method for** $f(x)=(x+2)(x-1)(x-3)$

Combining the shaded intervals gives the answer: $(-2;1)\cup(3;\infty)$.

4) Here, $f(x)=(x+1)^4(x-2)$. We put the points on the number axis: $x=-1$ and $x=2$, for which $f(x)=0$. The solution follows from Fig. 6.20.

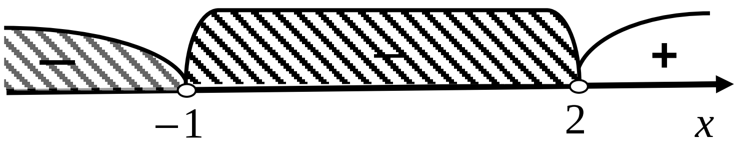


Fig. 6.20. **The spacing method for** $f(x)=(x+1)^4(x-2)$

So, the solution is $x\in(-\infty;-1)\cup(-1;2)$.

5) Here: $f(x)=(x+2)^2(x+1)^3(x-1)(x^2+3)$. Because $x^2+3>0$, for $x\in R$ the given inequality is equivalent to $(x+2)^2(x+1)^3(x-1)\le 0$. We solve this inequality by the spacing method (Fig. 6.21).

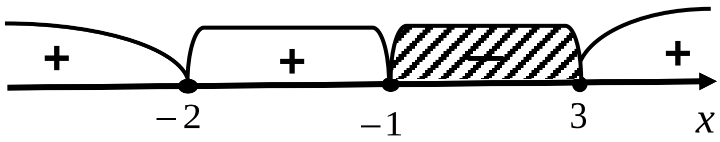


Fig. 6.21. **The spacing method for** $f(x)=(x+2)^2(x+1)^3(x-1)(x^2+3)$

Combining the obtained results, we have: $x\in\{-2\}\cup[-1;1]$.

Example 6.27. Solve the inequalities:

$$1)\ \frac{5x-6}{x+6}<1;\quad 2)\ \frac{x+7}{x-5}+\frac{3x+1}{2}\ge 0;\quad 3)\ \left|\frac{2}{x-4}\right|>1.$$

Solution:

1) Let us transform the inequality $\frac{5x-6}{x+6}<1$ to the form $f(x)<0$:

$$\frac{5x-6}{x+6}-1<0,\quad \frac{4x-12}{x+6}<0,\quad \frac{x-3}{x+6}<0.$$

One can apply the interval method to the last inequality. The result is shown in Fig. 6.22:

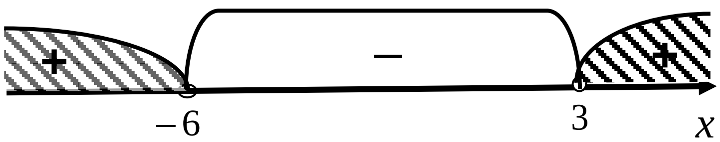


Fig. 6.22. **The spacing method for** $\frac{5x-6}{x+6}<1$

So, $x\in(-6;3)$.

2) Let us transform the inequality $\frac{x+7}{x-5}+\frac{3x+1}{2}\geq 0$ as follows:

$$\frac{2x+14+3x^2+x-15x-5}{2(x-5)}\geq 0,\quad \frac{3x^2-12x+9}{2(x-5)}\geq 0,\quad \frac{(x-3)(x-1)}{2(x-5)}\geq 0.$$

One can apply the spacing method to the last inequality (Fig. 6.23).

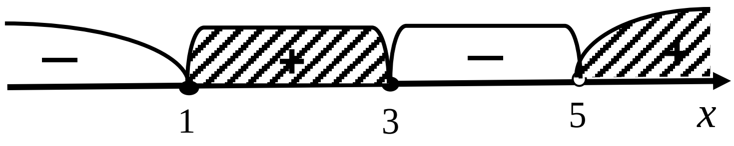


Fig. 6.23. **The spacing method for** $\frac{x+7}{x-5}+\frac{3x+1}{2}\geq 0$

So, $x\in[1;3]\cup(5;\infty)$.

3) The inequality $\left|\frac{2}{x-4}\right|>1$ is equivalent to the system of inequalities:

$$\left[\begin{array}{l}\frac{2}{x-4}>1,\\ \frac{2}{x-4}<-1;\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\frac{2}{x-4}-1>0,\\ \frac{2}{x-4}+1<0;\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\frac{6-x}{x-4}>0,\\ \frac{x-2}{x-4}<0.\end{array}\right.$$

Each inequality is solved by the method of intervals and the results obtained are combined (Fig. 6.24, 6.25).

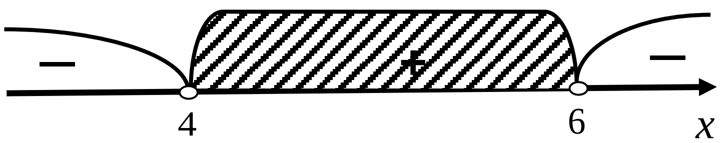


Fig. 6.24. **The method of intervals for** $\frac{6-x}{x-4}>0$

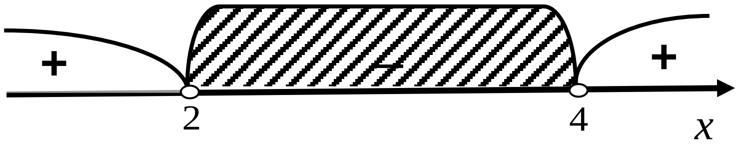


Fig. 6.25. **The method of intervals for** $\frac{x-2}{x-4}<0$

So, $x\in(2;3)\cup(4;6)$.

Example 6.28. Find the domains:

1) $y=\sqrt{\frac{x+3}{5-x}}$; 2) $y=\sqrt{\frac{4-3x-x^2}{x+4}}$; 3) $y=\sqrt{\frac{x^2-5x+6}{x^2+6x+8}}$.

Solution:

1) The domain $D(y)$, where $y=\sqrt{\frac{x+3}{5-x}}$ can be found by solving the inequality: $\frac{x+3}{5-x}\geq 0$, which can be solved by the interval method (Fig. 6.26).

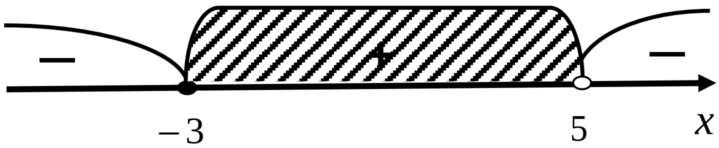


Fig. 6.26. **The method of intervals for** $y=\sqrt{\frac{x+3}{5-x}}$

So, $D(y)=[-3;5)$.

2) The domain $D(y)$, where $y=\sqrt{\frac{4-3x-x^2}{x+4}}$, can be found by solving the system of inequalities.

$\begin{cases}4-3x-x^2\geq 0,\\ x+4\neq 0;\end{cases}$ or $\begin{cases}4-3x-x^2\leq 0,\\ x\neq -4;\end{cases}$ or $\begin{cases}(x-1)(x+4)\leq 0,\\ x\neq -4.\end{cases}$

The inequality $(x-1)(x+4)\le 0$ is solved by the method of intervals, excluding the value of $x=-4$ (Fig. 6.27).

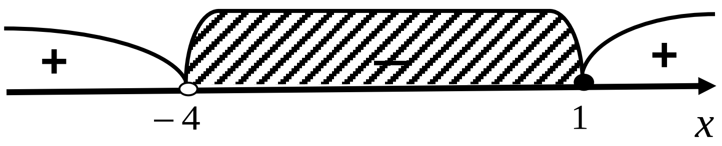


Fig. 6.27. **The method of intervals for** $y=\sqrt{\dfrac{4-3x-x^2}{x+4}}$

So, $D(y)=(-4;1]$.

3) For $y=\sqrt{\dfrac{x^2-5x+6}{x^2+6x+8}}$ $D(y)$ can be found from the inequality:

$$\frac{x^2-5x+6}{x^2+6x+8}\ge 0.$$

Let us factorize the numerator and denominator: the roots of the numerator are $\{2;3\}$, and the roots of the denominator are $\{-2;-4\}$. Get the fraction:

$$\frac{(x-2)(x-3)}{(x+2)(x+4)}\ge 0.$$

We apply the interval method taking into account that $x\ne -2$ and $x\ne -4$ (Fig. 6.28).

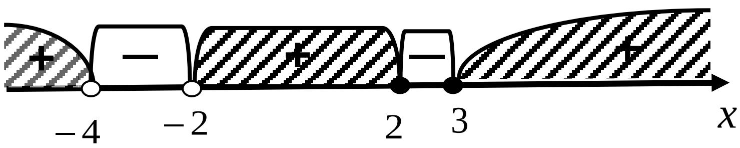


Fig. 6.28. **The method of intervals for** $y=\sqrt{\dfrac{x^2-5x+6}{x^2+6x+8}}$

So, $D(y)=(-\infty;-4)\cup(-2;2]\cup[3;\infty)$.

Tasks for individual work

6.29. Solve the inequalities

1) $\frac{2x-3}{3x-7}>0$; 2) $\frac{x-2}{x^2-9}\le 0$; 3) $(x-1)(3-x)(x-2)^2>0$;

4) $(x+1)(x-3)^2(x-5)^2(x-4)^2(x-2)\ge 0$; 5) $\frac{x-5}{x^2+5x-14}\ge 0$;

6) $\frac{x^2+2x-15}{x^2+1}<0$; 7) $\frac{3}{x-2}<1$; 8) $\frac{x-1}{x+3}>2$; 9) $\frac{x^2-x-6}{x^2+6x}\ge 0$;

10) $\frac{1}{x+2}<\frac{3}{x-3}$; 11) $\frac{x^2+6x-7}{|x+4|}<0$.

6.30. Find the domains:

1) $y=\sqrt{\frac{2x-3}{3x-7}}$; 2) $y=\sqrt{\frac{x}{x-5}-\frac{1}{2}}$; 3) $y=\sqrt{\frac{x^2+2x-3}{x^2+1}}$;

4) $y=\sqrt{4x-x^3}$; 5) $y=\frac{\sqrt{4-3x-x^2}}{x+4}$; 6) $y=\sqrt{x^2-x-20}+\sqrt{6-x}$.

6.8. Irrational inequalities

The inequality in which the variable is under the root sign is called irrational.

The main method for solving such inequalities is the method of raising both parts of the inequality to any power. It should be borne in mind that if $f(x)>0$ and $g(x)>0$ for $x\in R$, then

$$f(x)>g(x) \quad \Leftrightarrow \quad f^n(x)>g^n(x),\ n>1. \tag{6.3}$$

When raising both parts of irrational inequalities to an even power, the following equivalent statements are often used:

$$\sqrt[2n]{f(x)}<g(x) \quad \Leftrightarrow \quad \begin{cases} f(x)<g^{2n}(x), \\ f(x)\ge 0, \\ g(x)>0; \end{cases} \tag{6.4}$$

$$\sqrt[2n]{f(x)} > g(x) \quad \Leftrightarrow \quad \begin{cases} f(x) > g^{2n}(x), \\ g(x) \geq 0; \end{cases} \text{ or } \begin{cases} f(x) \geq 0, \\ g(x) < 0; \end{cases} \qquad (6.5)$$

$$\sqrt[2n]{f(x)} > \sqrt[2n]{g(x)} \quad \Leftrightarrow \quad \begin{cases} f(x) > g(x), \\ g(x) \geq 0. \end{cases} \qquad (6.6)$$

Example 6.31. Solve the inequality:

$$\sqrt{6-x} < 3x-4.$$

Solution: This inequality according to statement (6.4) is equivalent to the system of inequalities:

$$\begin{cases} 6-x < (3x-4)^2, \\ 3x-4 > 0, \\ 6-x \geq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 9x^2 - 23x + 10 > 0, \\ x > \dfrac{4}{3}, \\ x \leq 6. \end{cases} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} 9x^2 - 23x + 10 > 0, \\ \dfrac{4}{3} < x \leq 6. \end{cases} \Leftrightarrow \quad \begin{cases} x < \dfrac{5}{9}, x > 2, \\ \dfrac{4}{3} < x \leq 6. \end{cases}$$

This square inequality is solved by the method of intervals (Fig. 6.29).

$$9x^2 - 23x + 10 = 0 \quad x_1 = \frac{5}{9}, x_2 = 2, \qquad 9\left(x - \frac{5}{9}\right)(x-2) > 0.$$

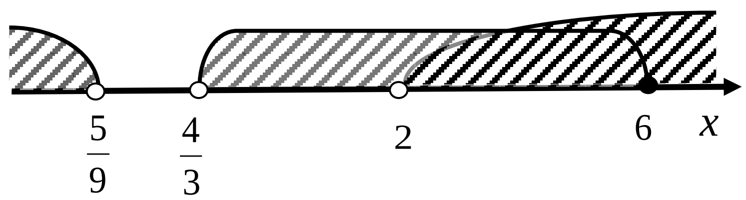


Fig. 6.29. **The method of intervals for** $\sqrt{6-x} < 3x-4$

So, $x \in (2;6]$.

Example 6.32. Solve the inequality:

$$\sqrt{x+7} > 2x-1.$$

Solution: This is the case corresponding to statement (6.5), i.e. this inequality is equivalent to the system of inequalities:

$$\sqrt{x+7} > 2x-1 \quad \Leftrightarrow \quad \begin{cases} x+7 > (2x-1)^2, \\ 2x-1 \ge 0; \end{cases} \text{ or } \begin{cases} x+7 \ge 0, \\ 2x-1 < 0; \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} 4x^2-5x-6<0, \\ x \ge \frac{1}{2}; \end{cases} \text{ or } \begin{cases} x \ge -7 \\ x < \frac{1}{2}; \end{cases} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} (x-2)\left(x+\frac{3}{4}\right) < 0, \\ x \ge \frac{1}{2}; \end{cases} \text{ or } \begin{cases} x \ge -7, \\ x < \frac{1}{2}; \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} -\frac{3}{4} < x < 2, \\ x \ge \frac{1}{2}; \end{cases} \text{ or } -7 \le x < \frac{1}{2}$$

$$\Leftrightarrow \quad \frac{1}{2} \le x < 2 \text{ or } -7 \le x < \frac{1}{2} \quad \Leftrightarrow \quad \left[\begin{array}{l} \frac{1}{2} \le x < 2; \\ -7 \le x < \frac{1}{2}. \end{array} \right.$$

The graphic solution of the last set of inequalities is presented in Fig. 6.30. There is a union of numeric spaces $[\frac{1}{2};2)$ and $[-7;\frac{1}{2})$.

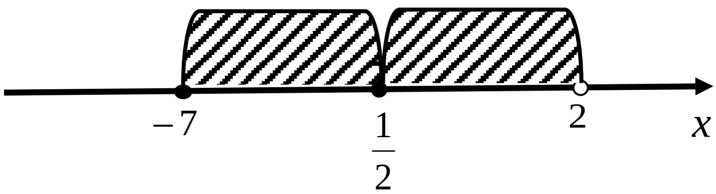


Fig. 6.30. **The method of intervals for** $\sqrt{x+7} > 2x-1$

So, $x \in [-7;\frac{1}{2}) \cup [\frac{1}{2};2) = [-7;2)$.

Each of these inequalities can be solved by the method of intervals. Consider this method in the following example.

Example 6.33. Solve the inequality:

$$\sqrt{x+3}-\sqrt{x-1}>\sqrt{2x-1}.$$

Solution: We write this inequality in the form:

$$\sqrt{x+3}-\sqrt{x-1}-\sqrt{2x-1}>0.$$

1. Let $f(x)=\sqrt{x+3}-\sqrt{x-1}-\sqrt{2x-1}$. Let us find the domain of $f(x)$:

$$D(f): \begin{cases} x+3\geq 0, \\ x-1\geq 0, \\ 2x-1\geq 0 \end{cases} \Leftrightarrow \begin{cases} x\geq -3 \\ x\geq 1, \ , \\ x\geq \frac{1}{2}. \end{cases} \Leftrightarrow \quad x\geq 1.$$

So, $D(f)=[1;\infty)$.

2. Let us solve the equation $\sqrt{x+3}-\sqrt{x-1}-\sqrt{2x-1}=0$:

$$x+3-2\sqrt{(x-1)(x+3)}+x-1=2x-1; \quad 2\sqrt{x^2+2x-3}=3;$$
$$4\left(x^2+2x-3\right)=9; \quad 4x^2+8x-21=0,$$

wherefrom: $x_1=\frac{3}{2}$; $x_2=-\frac{7}{2}$. However, $x=-\frac{7}{2}\notin D(f)$ but $x=\frac{3}{2}$ is the root of the equation to be solved.

Next, we solve the inequality $f(x)>0$ by the interval method (Fig. 6.31). We find the sign of $f(x)$ on intervals $[1;\frac{3}{2})$ and $[\frac{3}{2};\infty)$.

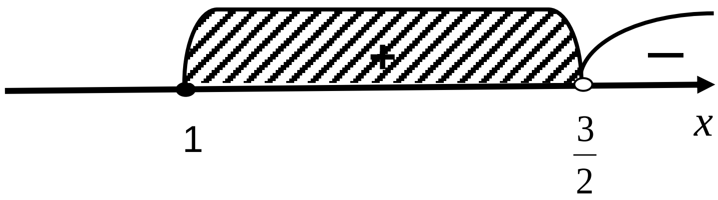


Fig. 6.31. **The method of intervals for** $\sqrt{x+3}-\sqrt{x-1}>\sqrt{2x-1}$

So, $x\in[1;\frac{3}{2})$.

The task for individual work

6.34. Solve the inequalities:

1) $\sqrt{3x-5}>7$; 2) $\sqrt{3x+1}>-5$; 3) $\sqrt{x}>\sqrt{2x-1}$;

4) $\sqrt{x+1}>\sqrt{x-1}$; 5) $\sqrt{4+3x-x^2}<2$; 6) $\sqrt{\frac{x+1}{2-x}}<3$;

7) $\sqrt{\frac{3x-1}{2-x}}>1$; 8) $\frac{\sqrt{x+4}}{1-x}<1$; 9) $\sqrt{2+x}>x$;

10) $\sqrt{x^2+1}>x-1$; 11) $\sqrt{2x-1}<x-2$; 12) $\sqrt{2-x}>x$;

13) $3\sqrt{x}-\sqrt{x+3}>1$; 14) $\sqrt{x+3}+\sqrt{x+15}<6$.

Questions for self-assessment

1. What are the algebraic inequalities?
2. What properties of numerical inequalities do you know?
3. What are the linear inequalities?
4. Inequalities with the sign of the module.
5. What are the quadratic inequalities?
6. What is the method of intervals?
7. What are the irrational inequalities?

7. Progressions

7.1. Arithmetic progression

A numerical sequence is a set of values, each of which has its own number $n \ (n=1.2,3,...)$, that is, a numerical sequence is a function of the natural argument.

A numerical sequence is most often denoted through x_n or $x_1, x_2,..., x_n$, where $n \in N$ and x_n is the common term of the sequence.

The arithmetic progression is the number sequence, each member of which, starting from the second, is equal to the previous member, to which the same number d is added. The number d is called the difference of the progression.

An arithmetic progression is indicated by (a_n) or $a_1, a_2,..., a_n,...$. From the definition of the progression it follows that

$$d = a_2 - a_1 = a_3 - a_2 = ... = a_{n+1} - a_n = ...$$

To set an arithmetic progression (a_n), it is enough to know its first term a_1 and the difference d. If $d > 0$, the progression is called increasing, if $d < 0$, the progression is called decreasing.

Note the two main properties of the arithmetic progression.

1. Each member of an arithmetic progression, starting with the second, is the arithmetic average of the previous and subsequent members:

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}, \quad n \in N, n \geq 2. \tag{7.1}$$

This property is not only necessary but also sufficient for the numbers a_{n-1}, a_n, a_{n+1} to form an arithmetic progression, that is, if $a_{n+1} - a_n = a_n - a_{n-1}$, then the numbers a_{n-1}, a_n, a_{n+1} are members of an arithmetic progression.

2. If an arithmetic progression has a finite number of its members, then the sums of the members equidistant from the ends of the progression are equal to each other and equal to the sum of the extreme members, that is,

$$a_1 + a_n = a_2 + a_{n+1} = a_3 + a_{n-2} = ... \tag{7.2}$$

The formula for the common term of an arithmetic progression (a_n) is:

$$a_n = a_1 + (n-1)d\,. \tag{7.3}$$

The sum of the first n members of an arithmetic progression can be found by the formulas:

$$S_n = \frac{a_1 + a_2}{2} n \tag{7.4}$$

or

$$S_n = \frac{2a_1 + d(n-1)}{2} n\,. \tag{7.5}$$

Example 7.1. Find the sum of the first twelve members of the arithmetic progression, if it is known that the sum of the third, fifth, fifteenth and seventeenth members is equal to 224.

Solution: According to the condition, it is necessary to find S_{19} if it is known that

$$a_3 + a_5 + a_{15} + a_{17} = 224\,.$$

According to the formula (7.4),

$$S_{19} = \frac{a_1 + a_{19}}{2} \cdot 19\,.$$

According to the formula (7.2),

$$a_1 + a_{19} = a_3 + a_{17} = a_5 + a_{15} = 112\,.$$

Then,

$$S_{19} = \frac{112}{2} \cdot 19 = 1064\,.$$

Answer: $S_{19} = 1064$.

Example 7.2. Find the arithmetic progression in which the sum of the first and ninth members is 6, and the product of the third and seventh members is 8.

Solution: By condition, $a_1 + a_9 = 6; a_3 \cdot a_7 = 8$. Using (7.3), we get the system:

$$\begin{cases} a_1 + (a_1 + 8d) = 6, \\ (a_1 + 2d) \cdot (a_1 + 6d) = 8. \end{cases}$$

Solving this simple system by substitution, we get:

$$d = \frac{1}{2}; a_1 = 1, \text{ or } d = -\frac{1}{2}; a_1 = 5.$$

Answer: $\left(1; \frac{3}{2}; 2; \ldots\right)$ or $\left(5; \frac{9}{2}; 4; \ldots\right)$.

Example 7.3. In an arithmetic progression of 20 members, the sum of even-numbered members is 250, and the sum of odd-numbered members is 220. Find the progression.

Solution: Obviously, the members of the desired arithmetic progression with even numbers $a_2, a_4, \ldots, a_{20}, \ldots$, as well as the members of odd numbers $a_1, a_3, \ldots, a_{19}, \ldots$, form the arithmetic progression. Each of them has 10 members and a difference $2d$, where d is the difference of the initial progression. S_{even} and S_{odd} can be found by the formula (7.5):

$$S_{even} = \frac{a_1 + a_{20}}{2} \cdot 10 = \frac{2a_1 + 20d}{2} \cdot 10 = (a_1 + 10d) \cdot 10.$$

$$S_{odd} = \frac{a_1 + a_{19}}{2} \cdot 10 = \frac{2a_1 + 18d}{2} \cdot 10 = (a_1 + 9d) \cdot 10.$$

By the condition of the problem, the system of equations can be derived:

$$\begin{cases} (a_1 + 10d)10 = 250, \\ (a_1 + 9d) \cdot 10 = 220. \end{cases}$$

After simplification, the system can be easily solved: $a_1 = -5;\ d = 3$.

Answer: $(-5;-2;1;\mathbf{...})$.

Example 7.4. At what values of α do the next three numbers

$$\frac{2}{3}\sin(\pi-\alpha),\ \sin\frac{\pi}{6}\ \text{ and }\ \frac{4}{3}\cos\left(\frac{\pi}{2}-\alpha\right)$$

form an arithmetic progression?

Solution: These numbers will be consecutive members of an arithmetic progression if the following equality holds:

$$\frac{2}{3}\sin(\pi-\alpha)+\frac{4}{3}\cos\left(\frac{\pi}{2}-\alpha\right)=2\sin\frac{\pi}{6}.$$

We solve this equation:

$$\frac{2}{3}\sin(\pi-\alpha)+\frac{4}{3}\cos\left(\frac{\pi}{2}-\alpha\right)=2\cdot\frac{1}{2}\Rightarrow\sin\alpha=\frac{1}{2}\Rightarrow$$

$$\alpha=(-1)^n\frac{\pi}{6}+\pi n,\ n\in Z.$$

Answer: $\left\{(-1)^n\frac{\pi}{6}+\pi n\right\}$.

Example 7.5. Find the sum of all three-digit numbers that are multiples of three.

Solution: Numbers that are multiples of three, – 102, 105, 108, ..., 999 – form an arithmetic progression in which $a_1 = 102;\ d = 3,\ \ a_n = 999$. We find $n =$ 300 according to formula (7.3). Then the sum of all three-digit numbers that are multiples of three is equal to

$$S_n = \frac{102+999}{2}\cdot 300 = 165\ 150.$$

Answer: $165\ 150$.

Example 7.6. Solve the equation:

$$\frac{x-1}{x^2}+\frac{x-2}{x^2}+\frac{x-3}{x^2}+\ldots+\frac{1}{x^2}=\frac{7}{15},$$

where x is a natural number.
Transform the equation to the form:

$$\frac{(x-1)+(x-2)+(x-3)+\ldots+1}{x^2}=\frac{7}{15}.$$

The numerator of this fraction is the sum of the members of the arithmetic progression in which $a_1=1, a_n=x-1, n=x-1$. We find this sum by the known formula (7.4), then the equation takes the form:

$$\frac{1+(x-1)}{2}\cdot(x-1)=\frac{7}{15}x^2, \text{ or } 15\cdot\left(x^2-x\right)=14x^2,$$

then:

$$x^2-15x=0,$$

wherefrom: $x=15$ ($x\neq 0$ by the condition).

Answer: 15.

Example 7.7. The sum of three numbers that are consecutive members of an arithmetic progression is 2, and the sum of their squares is 14/9. Find these numbers.

Solution: Let the numbers a_1, a_2, a_3 form an arithmetic progression. By the condition of the problem, we write the system of equations:

$$\begin{cases} a_1+a_2+a_3=2; \\ a_1^2+a_2^2+a_3^2=\dfrac{14}{9}. \end{cases}$$

According to formula (7.1), we have: $\frac{a_1 + a_3}{2} = a_2$. Then, from the first equation we have:

$$3a_2 = 2;\ a_2 = \frac{2}{3}.$$

The system takes the form:

$$\begin{cases} a_1 + a_3 = \frac{4}{3}; \\ a_1^2 + a_3^2 = \frac{10}{9}. \end{cases}$$

Solving the system by the method of, for example, substitutions, we get:

$a_1 = \frac{1}{3}, a_3 = 1$ or $a_1 = 1, a_3 = \frac{1}{3}$.

Answer: $\left(\frac{1}{3}; \frac{2}{3}; 1\right)$ and $\left(1; \frac{2}{3}; \frac{1}{3}\right)$.

7.2. Geometric progressions

A geometric progression refers to such a numerical sequence, each member of which, starting with the second, is equal to the previous one, multiplied by the same nonzero number q. The number q is called the denominator of the progression.

A geometric progression is denoted by (b_n) or $b_1, b_2, ..., b_n, ...$. From the definition of the progression it follows that

$$q = \frac{b_2}{b_1} = \frac{b_2}{b_1} = ... = \frac{b_{n+1}}{b_n} = ...$$

To set a geometric progression (b_n), it is enough to know its first term b_1 and the denominator q. When $q > 0\ (q \neq 1)$, a progression is called monotonously decreasing or monotonously increasing depending on the sign of b_1. If $q=1$, then all its members are equal to each other, and the progression is a constant sequence. Such a progression is usually not considered.

A geometric progression, like an arithmetic one, has two main properties.

1. In a geometric progression with positive members, each of its members, starting from the second, is the geometric mean of the previous and next members, that is, for $n \geq 2$:

$$b_n = \sqrt{b_{n-1} \cdot b_{n+1}} \ (n \in N). \tag{7.6}$$

Equality (7.6) is tantamount to the equality

$$b_n^2 = b_{n-1} \cdot b_{n+1}. \tag{7.7}$$

The fulfillment of this condition is not only necessary but also sufficient for the numbers b_{n-1}, b_n, b_{n+1} to form a geometric progression, that is, if $b_n^2 = b_{n-1} \cdot b_{n+1}$, then the numbers b_{n-1}, b_n, b_{n+1} form a geometric progression.

2. If a geometric progression has a finite number of members, then the product of the terms that are equidistant from the ends of the progression is equal to the product of the extreme members, that is,

$$b_1 \cdot b_n = b_2 \cdot b_{n-1} = b_3 \cdot b_{n-2} = \ldots. \tag{7.8}$$

The formula for a common term of a geometric progression is:

$$b_n = b_1 \cdot q^{n-1}. \tag{7.9}$$

The sum of the first *n* members of a geometric progression can be found by the formula:

$$S_n = \frac{b_1 - b_n \cdot q}{1-q} \tag{7.10}$$

or

$$S_n = \frac{b_1\left(1-q^n\right)}{1-q}. \tag{7.11}$$

Example 7.8. The fourth term of the geometric progression is 24 more than the second term, and the sum of the second and third members is 6. Find this progression.

Solution: By the condition, $b_4 - b_2 = 24$**,** $b_2 + b_3 = 6$.

Let us apply formula (7.9) of the common term of a geometric progression and write the system:

$$\begin{cases} b_1 q^3 - b_1 q = 24, \\ b_1 q + b_1 q^2 = 6 \end{cases} \Rightarrow \begin{cases} b_1 q\left(q^2 - 1\right) = 24, \\ b_1 q(1+q) = 6. \end{cases}$$

Dividing the first equation by the second one, we get:

$$\frac{b_1 q\left(q^2 - 1\right)}{b_1 q(1+q)} = \frac{24}{6} \Leftrightarrow q - 1 = 4, q = 5.$$

wherefrom:

$$b_1 = \frac{6}{q(1+q)} = \frac{1}{5}.$$

Answer: $\frac{1}{5};1;5;25;\ldots$.

Example 7.9. In a geometric progression, the sum of the first and fifth members is 51, and the sum of the second and sixth members is 102. How many members of this progression must be taken in order for the sum to be 3069?

Solution: By the condition, $b_1 + b_3 = 51$, and $b_2 + b_6 = 102$. According formula (7.9), we write the system:

$$\begin{cases} b_1 + b_1 q^4 = 51 \\ b_1 q + b_1 q^5 = 102 \end{cases} \Leftrightarrow \begin{cases} b_1(1 + q^4) = 51, \\ b_1 q(1 + q^4) = 102. \end{cases}$$

Dividing the second equation by the first one, we get $q=2$, and then,

$$b_1=\frac{51}{1+16}=3.$$

By the condition, $S_n=3069$. Applying formula (7.11), we write the equation:

$$\frac{3\left(1-2^n\right)}{1-2}=3069,$$

wherefrom $2^n=1024$, and then: $n=10$.

Answer: 10.

Example 7.10. Find a geometric progression if the sum of its first three members is 21, and the product of these members is 64.

Solution: By the condition, $b_1+b_2+b_3=21$, $b_1\cdot b_2\cdot b_3=64$. Let us apply formula (7.9) of the common term of a geometric progression and write the system:

$$\begin{cases} b_1+b_1q+b_1q^2=21; \\ b_1^3q^3=64. \end{cases} \Leftrightarrow \begin{cases} b_1+b_1q+b_1q^2=21; \\ b_1q=4. \end{cases} \Leftrightarrow \begin{cases} b_1+b_1q^2=17, \\ b_1q=4. \end{cases}$$

$$\begin{cases} \dfrac{b_1+b_1q^2}{b_1q}=\dfrac{17}{4}, \\ b_1q=4. \end{cases} \Leftrightarrow \begin{cases} 4\left(1+q^2\right)=17q \\ b_1q=4. \end{cases} \Leftrightarrow \begin{cases} 4q^2-17q+4=0 \\ b_1=\dfrac{4}{q} \end{cases} \Leftrightarrow$$

$$\begin{cases} \left[\begin{array}{l} q_1=4, \\ q_2=\dfrac{1}{4}, \end{array}\right. \\ b_1=\dfrac{4}{q}. \end{cases}$$

Answer: For $q=4$, $b_1=1$, and for $q=\frac{1}{4}$, $b_1=16$.

Example 7.11. The equation $x^3 + 3x^2 - 6x + a = 0$ has three different real roots that form a geometric progression. Find the value a and the roots of the equation.

Solution: Let x_1, x_2 and x_3 be the roots of the equation. They form a geometric progression, $x_2 = x_1 q, x_3 = x_1 q^2$. According to the Vieta's theorem for a cubic equation,

$$x_1 + x_2 + x_3 = -3; \;\; x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = -6; \;\; x_1 \cdot x_2 \cdot x_3 = -a.$$

So, we have a system of equations:

$$\begin{cases} x_1\left(1 + q + q^2\right) = -3, \\ x_1^2\left(q + q^2 + q^3\right) = -6, \\ x_1^3 q^3 = -a. \end{cases}$$

Dividing the second equation by the first one, we get: $x_1 q = 2$. that is $x_2 = 2/q$. Dividing further the first equation by $x_1 q = 2$, we get:

$$\frac{1 + q + q^2}{q} = -\frac{3}{2} \Leftrightarrow 2q^2 + 5q + 2 = 0,$$

wherefrom: $q = -2$ and $q = -\frac{1}{2}$. Then, for $q = -2$, we have: $x_1 = -1, x_3 = -4$; and for $q = -\frac{1}{2}$ $x_1 = -4, x_3 = 1.$ So, the roots are: $-1; 2; -4; a = -8.$

Answer: $a = -8;$ $(-1; 2; -4)$.

Example 7.12. Find three numbers that form a decreasing geometric progression, if the sum of these numbers is 74, and the sum of their squares is 1924.

Solution: By the condition, $b_1+b_2+b_3=74$, $b_1^2+b_2^2+b_3^2=1924$. We apply formula (7.9) and get a system with two unknowns, b_1 and q:

$$\begin{cases} b_1+b_1q+b_1q^2=74; \\ b_1^2+b_1^2q^2+b_1^2q^4=1924. \end{cases} \Leftrightarrow \begin{cases} b_1\left(1+q+q^2\right)=74; \\ b_1^2\left(1+q^2+q^4\right)=1924. \end{cases}$$

Dividing the first equation by the second one, we get:

$$\frac{1+q+q^2}{b_1\left(1+q^2+q^4\right)}=\frac{74}{1924} \Leftrightarrow \frac{1+q+q^2}{b_1\left(1+q+q^2\right)\left(1-q+q^2\right)}=\frac{1}{26} \Leftrightarrow$$

$$b_1\left(1-q+q^2\right)=26.$$

Then:

$$\begin{cases} b_1\left(1-q+q^2\right)=26, \\ b_1\left(1+q+q^2\right)=74. \end{cases}$$

Again, we divide the first equation by the second one:

$$\frac{1-q+q^2}{\left(1+q+q^2\right)}=\frac{26}{74} \Leftrightarrow 12q^2-25q+12=0,$$

wherefrom, $q=\frac{3}{4}$ or $q=\frac{4}{3}$. By the condition of the problem, it is necessary to obtain a decreasing progression, because $q=\frac{4}{3}>1$ does not fit. Then we have the solution:

$$q=\frac{3}{4},\ b_1=\frac{74}{1+\frac{3}{4}+\frac{9}{16}}=32,\ b_2=24,\quad b_3=18.$$

Answer: $32;24;18.$

Example 7.13. The product of the first three members of the geometric progression is 64, and the sum of their cubes is 584. Find the progression.

Solution: By the condition, $b_1 \cdot b_2 \cdot b_3 = 64$, $b_1^3 + b_2^3 + b_3^3 = 584$. Applying formula (7.9) of the common term of a geometric progression, we obtain the system of equations:

$$\begin{cases} b_1 \cdot b_1 q \cdot b_1 q^2 = 64, \\ b_1^3 + b_1^3 q^3 + b_1^3 q^6 = 584 \end{cases} \Rightarrow \begin{cases} b_1^3 q^3 = 64, \\ b_1^3 (1 + q^3 + q^6) = 584. \end{cases}$$

Divide the second equation by the first one:

$$\frac{1 + q^3 + q^6}{q^3} = \frac{584}{64}.$$

Let $q^3 = t$, then we have the equation $8t^2 - 65t + 8$, from which we find:

$$t = 8 \text{ and } t = \frac{1}{8}.$$

So, $q^3 = 8$ or $q^3 = \frac{1}{8}$, that is: $q = 2$ or $q = \frac{1}{2}$. For $q = 2$, $b_1 = 2$; or for $q = \frac{1}{2}$, $b_1 = 8$.

Answer: $(2,4,8)$ or $(8,4,2)$.

7.3. Infinitely decreasing geometric progressions

An infinitely decreasing geometric progression is a geometric progression (b_n) in which the denominator $|q| < 1$.

The sum of an infinitely decreasing geometric progression is the limit of the sum of its first n terms for $n \to \infty$. Let us denote the sum of an infinitely decreasing geometric progression by S, for which the following formula holds:

$$S = \lim_{n \to \infty} S_n = \frac{b_1}{1 - q}. \qquad (7.12)$$

Example 7.14. The sum of the members of an infinitely decreasing geometric progression is 18, and the sum of its first four members is equal to $16\frac{7}{8}$. Find the first member of this progression.

Solution: By the condition, $S = 18$, $S_4 = 16\frac{7}{8}$. Using the formulas, we obtain the system of equations:

$$\begin{cases} \frac{b_1}{1-q} = 18, \\ \frac{b_1(1-q^4)}{1-q} = \frac{135}{8} \end{cases} \quad \begin{cases} 1-q^4 = \frac{15}{16}, \\ b_1 = 18(1-q) \end{cases} \Leftrightarrow \begin{cases} q^4 = \frac{1}{16}, \\ b_1 = 18(1-q). \end{cases}$$

So, we have: $q = \frac{1}{2}$ or $q = -\frac{1}{2}$. For $q = \frac{1}{2}$ we get $b_1 = 9$; for $q = -\frac{1}{2}$ we get $b_1 = 27$.

Answer: 9 or 27.

Example 7.15. The sum of the members of an infinitely decreasing geometric progression is 16, and the sum of its first four members is $153\,{}^3/_5$.

Find the denominator and the fourth member of this progression.

Solution: By the condition, we have:

$$\begin{cases} \frac{b_1}{1-q} = 16, \\ b_1^2 + (b_1 q)^2 + (b_1 q^2)^2 + \ldots = 153\frac{3}{5} \end{cases} \Leftrightarrow \begin{cases} \frac{b_1}{1-q} = 16, \\ b_1^2(1+q^2+q^4+\ldots) = \frac{768}{5} \end{cases} \Leftrightarrow$$

$$\begin{cases} \frac{b_1}{1-q} = 16, \\ \frac{b_1^2}{1-q^2} = \frac{768}{5}. \end{cases}$$

Dividing the second equation by the first one, we get:

$$\begin{cases} \dfrac{b_1}{1+q} = \dfrac{48}{5}, \\ \dfrac{b_1}{1-q} = 16. \end{cases} \Leftrightarrow \begin{cases} b_1 = 16(1-q), \\ \dfrac{1-q}{1+q} = \dfrac{3}{5} \end{cases} \Leftrightarrow \begin{cases} q = \dfrac{1}{4}, \\ b_1 = 12. \end{cases}$$

So, $b_4 = b_1 q^3 = 12 \cdot \dfrac{1}{64} = \dfrac{3}{16}$.

Answer: $q = \dfrac{1}{4}, b_4 = \dfrac{3}{16}$.

Example 7.16. Determine the sum of the members of the infinitely decreasing geometric progression

$$x^{-1} + x^{-2} + x^{-3} + ..., \qquad |x| > 1.$$

Solution: This amount is the sum of the members of the infinitely decreasing geometric progression in which

$$b_1 = x^{-1} = \frac{1}{x}; \; q = \frac{1}{x} \; (|q| < 1).$$

Using the formula $S = \dfrac{b_1}{1-q}$, we have:

$$S = \frac{1}{x} : \left(1 - \frac{1}{x}\right) = \frac{1}{x} : \frac{x-1}{x} = \frac{1}{x-1}.$$

Answer: $\dfrac{1}{x-1}$.

7.4. Combined problems on arithmetic and geometric progressions

Combined problems on progressions consider such problems that use the properties of both arithmetic and geometric progressions.

Example 7.17. Three numbers, the sum of which is equal to 21, form an arithmetic progression. If we subtract 1 from the second number, and add 1 to the third one, we obtain a geometric progression. Find these numbers.

Solution: Let the numbers x, y, z form an arithmetic progression, and then according to the condition the numbers $x, y-1, z+1$ form a geometric progression. According to the properties of these progressions and the condition of the problem, we have the system of equations:

$$\begin{cases} x+y+z=21, \\ x+z=2y, \\ x(z+1)=(y-1)^2. \end{cases} \Leftrightarrow \begin{cases} 3y=21, \\ x+z=2y, \\ x(z+1)=(y-1)^2. \end{cases} \Leftrightarrow \begin{cases} y=7 \\ x+z=14, \\ x(z+1)=36. \end{cases}$$

From the last two equations, the substitution $z=x-14$ yields the equation $x^2-15x+36=0$, from which we have: $x_1=3, x_2=12$. Then, $z_1=11, z_2=2$.

Answer: $(3;7;11)$ or $(12;7;2)$.

Example 7.18. The first and third members of an arithmetic progression, respectively, are equal to the first and third members of a geometric progression. If the second term of the arithmetic progression is reduced by 1/4, then we get the second term of the geometric progression. The first member of the arithmetic progression is 2. Find its second and third members.

Solution: By the condition, the numbers $2, y, z$ form an arithmetic progression, and the numbers $2, y-\frac{1}{4}, z$ form a geometric one. According to the properties of these progressions, we have the system of equations:

$$\begin{cases} z+2=2y, \\ \left(y-\frac{1}{4}\right)^2=2z, \end{cases}$$

which easily reduces to the quadratic equation $16y^2-72y+65=0$, from which we get: $y=\frac{5}{4}$ or $y=\frac{13}{4}$.

For $y=\frac{5}{4}$, $z=2\cdot\frac{5}{4}-2=\frac{1}{2}$; for $y=\frac{13}{4}$ $z=2\cdot\frac{13}{4}-2=\frac{9}{2}$.

Answer: $\left(\frac{5}{4};\frac{1}{2}\right)$ or $\left(\frac{13}{4};\frac{9}{2}\right)$.

Example 7.19. Three numbers x, y, z form a geometric progression in this order, and numbers $x, 2y, 3z$ form an arithmetic progression in this order. Find the denominator of the geometric progression, which is not equal to 1.

Solution: The numbers x, y, z form the geometric progression: $y=xq,\ z=xq^2$. The numbers $x, 2y, 3z$ form an arithmetic progression, so $4y=x+3z$ or, after substituting $y=xq,\ z=xq^2$, we have the equation: $4xq=x+3xq^2$. By the condition, $x\neq 0$. Dividing by x, we get the equation $3q^2-4q+1=0$, from which we have: $q=\frac{1}{3}$ or $q=1$. By the condition, $q\neq 1$. Then, $q=\frac{1}{3}$.

Answer: $q=\frac{1}{3}$.

Tasks for individual work

The arithmetic progression

7.20. Find the first term and the difference of the arithmetic progression in which it is known that:

a) $\begin{cases} a_2+a_8=10, \\ a_3+a_{14}=31. \end{cases}$ b) $\begin{cases} S_5-S_2-a_5=0.1, \\ S_4+a_7=0.1. \end{cases}$ c) $\begin{cases} S_4=9, \\ S_6=22.5. \end{cases}$

7.21. Find the arithmetic progression, if $a_2 + a_5 - a_3 = 10$, and $a_1 + a_6 = 7$.

7.22. Find the arithmetic progression, if $a_1 + a_4 = 16$, and $a_2 \cdot a_3 = 60$.

7.23. In the arithmetic progression $a_1 + a_4 = -4$, and $a_2 \cdot a_3 = 32$. Find the sum of the first six members of this progression if it is known that all its members are integers.

7.24. Find the tenth member of an arithmetic progression if the sum of its n members is equal to $3n^2 - 2n$.

7.25. Find the sum of 20 members of the arithmetic progression, in which $a_1 = 2$, $a_7 = 20$.

7.26. The sum of the third and ninth members of an arithmetic progression is 8. Find the sum of the first 11 members of this progression.

7.27. How many members of an arithmetic progression should be taken for their sum to be 91, if $a_3 = 9$, $a_7 - a_2 = 20$?

7.28. Three numbers, the sum of which is equal to 21, and the sum of their squares is equal to 155, form an arithmetic progression. Find these numbers.

7.29. Three numbers form an arithmetic progression. Their sum is 4, and the sum of their cubes is 3. Find these numbers.

7.30. Solve the equation: $1 + 7 + 13 + \dots + x = 280$.

7.31. Solve the equation: $(x+1) + (x+4) + \dots + (x+28) = 155$.

7.32. Find the sides of the triangle if they are integers and form an arithmetic progression. It is known that the perimeter of the triangle is equal to 15.

The geometric progression

7.33. Find the geometric progression, if $b_3 - b_1 = 16$.

7.34. Find the four numbers that form a geometric progression in which the third term is greater than the first one by 9, and the second term is greater than the fourth one by 18.

7.35. Find the first member and the number of the members of the geometric progression in which $b_7 - b_5 = 48$, $b_6 + b_5 = 48$, $S_n = 1023$.

7.36. In a geometric progression $S_6 = 1820$, $q = 3$. Find b_1 and b_5.

7.37. Find the number of the members of a geometric progression, if it is known that $b_1 = 3, b_2 = 12$ and $b_n = 3072$.

7.38. In a geometric progression, it is known that $b_4 = b_2 + 243, b_2 + b_3 = 6$. Find this progression.

7.39. The sum of the first and third members of a geometric progression is 26, and the sum of the first three members is 31. Find the seventh member of this progression.

7.40. How much will be a four-year bank deposit of 1000 UAH, on the basis that it increases annually by 2 %?

The infinitely decreasing geometric progression

7.41. The sum of an infinitely decreasing geometric sequence is 12.5, and the sum of its first members is 12. Find the progression.

7.42. The sum of the infinitely decreasing geometric progression is 243, and the sum of its first 5 members is 275. Find the progression.

7.43. The first term of an infinitely decreasing geometric progression is equal to $\sqrt{3}$, and the second term is equal to $\frac{2}{\sqrt{3}+1}$**.** Find the progression.

7.44. The sum of the members of an infinitely decreasing geometric progression is equal to 9, and the sum of the squares of its members is 40.5. Find the progression.

7.45. The sum of an infinitely decreasing geometric progression is 4, and the sum of the cubes of its members is 64/7. Find the sixth member of this progression.

7.46. Solve the equation: $\frac{1}{3x} + x + x^2 + \mathbf{...} + x^n + \mathbf{...} = \frac{3}{2}$, where $|x| < 1$.

7.47. Solve the equation: $2x + 1 + -x^2 + x^4 - \mathbf{......} = \frac{13}{6}$, where $|x| < 1$.

Mixed problems on arithmetic and geometric progressions

7.48. Find a three-digit number if its digits form a geometric progression, and the digits of the number that is less than the given one by 400 form an arithmetic progression.

7.49. The sum of the three members of an arithmetic progression is 54. If the second term is reduced by 9, and the third one by 6, then we get a geometric progression. Find the arithmetic progression.

7.50. The sum of the three members that make up a geometric progression is 65. If the smaller of them is reduced by 1, and the larger one by 19, then an arithmetic progression is formed. Find these numbers.

7.51. Find four integers if it is known that the first three of them form a geometric progression, and the last three form an arithmetic one, the sum of the extreme numbers is 21, and the sum of the averages is 18.

7.52. The numbers $5x-y, 2x+3y$ and $x+2y$ form an arithmetic progression, and the numbers $5x-y, 2x+3y$ and $(x-1)^2$ form a geometric progression. Find x and y, if $x \neq 0$ and $y \neq 0$.

Questions for self-assessment

1. What is the arithmetic progression?
2. What is the geometric progression?
3. What is the infinitely decreasing geometric progression?

Answers

1

1.48. 1) $2^3\cdot 3^3$; $2\cdot 3^4$; $2^4\cdot 3^2$; $3^2\cdot 5^2$; 2^9; $3^3\cdot 5^2$; 2^{10}.

2) $2^2\cdot 3\cdot 5$; $2^2\cdot 3^2\cdot 5$; $2^2\cdot 5\cdot 11$; $2\cdot 5^2\cdot 7$; $2^4\cdot 5^2$; $2^4\cdot 3\cdot 5^2$.

3) $5^3\cdot 7$; $3^4\cdot 5^2$; $2^3\cdot 3^3\cdot 11$; $3^4\cdot 7^2$.

1.49. 6; 25; 75; 360; 36; 40. **1.50.** 792; 2550; 288; 3180. **1.51.** 16. **1.52.** 1. **1.53.** 0,1. **1.54.** 2,5. **1.55.** $-26{,}5$. **1.56.** 27,5. **1.57.** 107. **1.58.** 6. **1.59.** 4,5. **1.60.** 1) 4; 2) $\frac{10}{11}$; 3) 12; 4) $\frac{2}{5}$; 5) $\frac{15}{8}$; 6) $\frac{1}{2}$. **1.61.** 576. **1.62.** 288. **1.63.** 152. **1.64.** 40. **1.65.** 80. **1.66.** 210. **1.67.** 150. **1.68.** 20%. **1.69.** 840. **1.70.** 1260 UAN. **1.71.** 6,25 kg. **1.72.** 116,2 kg. **1.73.** 5%. **1.74.** 50. **1.75.** 60. **1.76.** 10 %. **1.77.** 1,5 kg. **1.78.** 60 kg. **1.79.** 25. **1.80.** 0,118.

2

2.31. 1) $a^2+2ax+x^2$; 2) $\frac{t^2}{9}-\frac{4}{3}+\frac{4}{t^2}$; 3) $\frac{1}{16}a^2-ab+4b^2$; 4) $4a^{2k}-12a^2+$ $+9a^{4-2k}$; 5) $1-81a^4$; 6) $3^{8m}-2\cdot 3^{4m+6n}+3^{12n}$; 7) $\frac{1}{4}a^2-\frac{1}{6}ab+\frac{1}{9}b^2$.

2.32. 1) $5y^2(y-3)$; 2) $a^2(a^{10}-b)$; 3) $8^{2n-1}(1+8^2)$; 4) $3^{x-1}(2\cdot 3^2-4+3)$; 5) $(2b-3)(5a-3b-1)$; 6) $3ab(2x^2-3y^2)(2a-3)$. **2.33.** 1) $(a+b)(c+1)$; 2) $(x+y)(a-1)$; 3) $(3b^2-2c)(5c-16a)$; 4) $(a+b)(a^2-b)$; 5) (x^2+2x+8) (x^2+3x+8). **2.34.** $\frac{12(2a+5)(a+3)}{a-2}$. **2.35.** 1) $\frac{1}{5}$; 2) $\frac{15}{4}$. **2.36.** $\frac{x+2}{2}$.

2.37. $\frac{x+1}{(2y-x)(2x^2+y+2)}$. **2.38.** $\frac{1}{2ab(a+b)}$. **2.39.** $x(x-a)$. **2.40.** $\frac{1}{ab}$.

2.41. 1) $-2\sqrt{3}$; 2) $\sqrt{3}-\sqrt{2}$; 3) $2-\sqrt{3}$. **2.42.** $\frac{1}{4}$. **2.43.** 1. **2.44.** $\frac{1-a}{\sqrt{a}}$.

2.45. $\frac{1}{ab\sqrt[5]{a^3}}$.

3

3.12. 1) $x \in R \setminus \{2\}$; 2) $x \in R$; 3) $x \in (-\infty; 0{,}5]$; 4) $1 \le x < 4$; 5) $x \in R$; 6) $-2 < x < -1/3$; 7) $x \in (-\infty; 1] \cup [2; \infty)$; 8) $x \in (0; 5/2] \cup [3; +\infty)$; 9) $x \le 0$; 10) $x = 0$. **3.13.** 1) , 3), 4), 8) – even; 2), 5), 6) – odd; 7), 9) – general form. **3.17.** 1) $y = \frac{x-10}{10}$; 2) $y = \frac{x+2}{5}$; 3) $y = x^2$; 4) $y = \sqrt[3]{x}$.

4

4.10. $\{51\}$. **4.11.** $\left\{15\frac{2}{13}\right\}$. **4.12.** $\{\varnothing\}$. **4.13.** $R \setminus \{\pm 3\}$. **4.14.** $x = \frac{3a}{a-3}$, if $a \ne 3; x \in \varnothing$, if $a = 3$. **4.15.** $x = \frac{2}{a}$, if $a \ne 0; \pm 1$; $x \in R$, if $a = 1$; $x \in \varnothing$, if $a = 0$.

4.16. $x = \frac{1}{m}$, if $m \ne 0; 3$; $x \in R$, if $m = 3$; $x \in \varnothing$, if $m = 0$.

4.17. If $n = 0$, −3: $x \in \varnothing$, if $n = \frac{1}{2}$. **4.18.** $\{-5; 5\}$. **4.19.** $\{-1; 5\}$. **4.20.** $\{-8; 18\}$.

4.21. $\{-1\}$. **4.22.** $\left\{\frac{1}{3}\right\}$. **4.23.** $[-1; 1]$. **4.31.** $\{-1; -2102\}$. **4.32.** $\{7; -10\}$.

4.33. $\left\{-\frac{1}{2}; \frac{1}{2}\right\}$. **4.34.** $\left\{-\frac{1}{2}; 2\right\}$. **4.35.** $\{2; 3; 4\}$. **4.36.** $\varnothing$. **4.37.** $\{2\}$.

4.38. $(-\infty; -2] [2; +\infty)$. **4.39.** $(-\infty; 2] \cup [3; \infty)$. **4.40.** $\{\varnothing\}$. **4.41.** $\left\{\frac{1}{6}\right\}$.

4.42. $\left\{-\frac{1}{6}; \frac{1}{6}\right\}$. **4.43.** $\{3\}$. **4.44.** $[-2; 3]$. **4.45.** $\{5\}$. **4.46.** $\{30\} \{15\}$.

4.47. $\left\{-\frac{5}{2}; \frac{3}{2}\right\}$. **4.48.** $\{4\}$. **4.49.** $\left\{\frac{1}{2}; 1\right\}$. **4.50.** $\{2/3\}$. **4.56.** 96 m. **4.57.** 12500 kg.

4.58. 240 kg. **4.59.** 85. **4.60.** 24. **4.61.** 2100 UAH. **4.62.** 20 days and 30 days.

4.63. 8 h. **4.64.** 23 and 24. **4.65.** $\frac{3}{5}$. **4.66.** 10%. **4.67.** 30 rows.

4.68. 5 km/h and 3 km/h. **4.69.** 5 days. **4.70.** 2 h. **4.71.** 80 km/h and 60 km/h.

4.72. 24 km/h. **4.84**. $\{0; 5\}$. **4.85.** $\left\{\frac{1}{3}\right\}$. **4.86.** $\{1; 2; 3; 4\}$. **4.87.** $\left\{\frac{-1 \pm \sqrt{21}}{2}\right\}$.

4.88. $\{0\}$. **4.89.** $\left\{\frac{1}{2};2\right\}$. **4.90.** $\{0;3\}$. **4.91.** $\{-4;0;3;\}$. **4.92.** $\{0;1\}$. **4.93.** $\{-2;1;2;\}$.

4.94. $\{-2;1;\}$. **4.95.** $\left\{1;\frac{7\pm\sqrt{33}}{4}\right\}$. **4.96.** $\left\{2;4;\frac{-7\pm\sqrt{17}}{2}\right\}$. **4.97.** $\left\{3;3\pm2\sqrt{5}\right\}$.

4.98. $\left\{-1;0;\frac{-1\pm\sqrt{5}}{2}\right\}$. **4.99.** $\left\{1\pm\sqrt{3};\right\}$. **4.100.** $\left\{-4\pm\sqrt{5}\right\}$. **4.101**. $\left\{\frac{-5\pm\sqrt{21}}{6}\right\}$.

4.102. $\left\{\frac{-11\pm\sqrt{97}}{6}\right\}$. **4.103.** $\left\{3;\ 5;9\pm\sqrt{66};\right\}$. **4.104.** $\left\{\frac{3\pm\sqrt{5}}{2};\frac{-3\pm\sqrt{5}}{2}\right\}$.

4.105. $\left\{\frac{1\pm\sqrt{21}}{2};\right\}$. **4.106.** $\left\{-2;\frac{2}{3};\frac{4}{3};4\right\}$. **4.107.** $\left\{\frac{2}{7};\frac{-7\pm\sqrt{137}}{22}\right\}$. **4.124.** $\{2\}$.

4.125. $\{-1\}$. **4.126.** $\{-3;2\}$. **4.127.** $\{1\}$. **4.128.** $\{5\}$. **4.129.** $\{2\}$. **4.130.** $\{53\}$.

4.131. $\{4\}$. **4.132.** $\{20\}$. **4.133.** $\{-5;4\}$. **4.134.** $\left\{\frac{1}{3}\right\}$. **4.135.** $\{-3\}$.

4.136. $\{27;-8\}$. **4.137.** $\{8;27\}$. **4. 138.** $\{1\}$. **4.139.** $\left\{\pm2\sqrt{2}\right\}$. **4.140.** $\left\{\pm2\sqrt{2}\right\}$.

4.141. $\{64\}$. **4.142.** $\{64\}$. **4.143.** $\{-7;7\}$. **4.144.** $\{76\}$. **4.145.** $\{-4;2\}$.

4.146. $\left\{-\frac{1}{3};1\right\}$. **4.147.** $\{\pm0,5\}$. **4.148.** $\{-1\}$. **4.149.** $\{-1;7\}$. **4.150.** $\{7\}$.

4.151. $\left\{-\frac{3}{2};-\frac{2}{5};\frac{1}{3}\right\}$. **4.152.** $\{\varnothing\}$. **4.153.** $\{\varnothing\}$. **4.154.** $\{[0,3]\}$. **4.155.** $\{[2;\infty)\}$.

4.156. $\{8\}$. **4.157.** $\left\{-1;\frac{9}{16}\right\}$.

5

5.21. $a\neq\pm6$. **5.22.** $x=\frac{5}{3}-\frac{9}{3}y$. **5.23.** $(2;-3)$. **5.24.** $(4;1)$. **5.25.** $(2;1)$.

5.26. $(2;3)$. **5.27.** $(0;-1)$. **5.28.** $\{(-6;9),(0;-3),(2;1)\}$. **5.29.** $\{(0;3),(4;1)\}$.

5.30. $\left\{\left(-\frac{11}{19};\frac{23}{19}\right),(1;-1)\right\}$. **5.31.** $\left\{\left(\frac{1-2a^2}{1-a^2};\frac{a}{1-a^2}\right)\right\}$ if $a\neq\pm1$; $\varnothing$ if $a=\pm1$.

5.32. $\left\{\left(\frac{a^2+1}{a^2};\frac{a+1}{a^2}\right)\right\}$ if $a\neq0$; $\varnothing$ if $a=0$. **5.33.** $(1;-2)$. **5.34.** $(-1;2)$.

5.35. $(5;-2)$. **5.36.** $(1;-1)$. **5.37.** $(3;2)$. **5.38.** $\left(\frac{1}{3};\frac{2}{3}\right)$. **5.39.** $\left(\frac{a-1}{a+2};\frac{2}{a+2}\right)$ if $a\neq\pm2$; incompatible if $a=-2$; a countless set if $a=2$. **5.40.** $(0;0)$ if $a\neq0$ and $a\neq3$; a countless set if $a=0$ and $a=3$. **5.41.** $(1;2;-1)$. **5.42.** $(-1;1;0)$ **5.43.** $(2;0;-1)$. **5.44.** $(2;1;-1)$. **5.45.** $(2;3)$. **5.46.** $\{(1;4),(4;1)\}$. **5.47.** $\left\{(4;1),\left(-9;-\frac{9}{4}\right)\right\}$. **5.48.** $\{(2;8),(8;2),(-2;-8), (-8;-2)\}$. **5.49.** $\{(-1;-2),(-\sqrt{2};-\sqrt{2}),(1;2),(\sqrt{2};\sqrt{2})\}$. **5.50.** $\left\{(-3;-5),\left(-\frac{5}{3};-\frac{13}{3}\right),\left(\frac{5}{3};\frac{13}{3}\right),(3;5)\right\}$. **5.51.** $\{(1;2),(2;1),(-3;1),(1;-3)\}$. **5.52.** $\{(3;5),(5;3)\}$. **5.53.** $\{(1;81),(81;1)\}$. **5.54.** $(5/3;1/3)$. **5.55** $(1350;27)$. **5.56.** $(6;12)$. **5.57.** $(12;15)$. **5.58.** $\{4\}$. **5.59.** $(19;67)$. **5.60.** $(30;30)$.

6

6.10. 1) $(4;\infty)$; 2) $(-\infty;43)$; 3) $\varnothing$; 4) $(-\infty;\infty)$. **6.11.** 1) $x>\frac{6a-1}{a+3}, a>-3$; $x<\frac{6a-1}{a+3}, a<-3$; $x\in R, a=-3$; 2) $x<\frac{2-a}{a+1}, a>-1$; $x>\frac{2-a}{a+1}, a<-1$; $x\in R, a=-1$. **6.12.** 1) $(-\infty;2)$; 2) $(-\infty;1)$; 3) $(-2;\infty)$; 4) $(-7;7)$; 5) $(1{,}5;\infty)$; 6) $\varnothing$. **6.13.** 1) $(-\infty;1)\cup(3;\infty)$; 2) $(-\infty;10)$; 3) $(-7;\infty)$; 4) $(-\infty;\infty)$. **6.14.** 1) $[2;4)$; 2) $\left[\frac{1}{2};2\right)$; 3) $\left(\frac{5}{2};11\right)$; 4) $\left[-2;-\frac{3}{2}\right)$. **6.15.** 1) $(-6;-1)$; 2) $\left(-\infty;\frac{1}{2}\right]\cup\left[\frac{3}{2};\infty\right)$; 3) $\left(\frac{5}{3};3\right)$; 4) $(-\infty;-3)\cup(2;\infty)$; 5) $(4{,}5;\infty)$; 6) $(-\infty;-2)\cup(5;\infty)$. **6.16.** 1) $[1;\infty)$; 2) $[1;6]$; 3) $[-3;5)$; 4) $[-1;2]$; 5) $[-2;1)\cup(1;\infty)$. **6.23.** 1) $\left[1;\frac{4}{3}\right]$; 2) $(-\infty;-1]\cup[8;\infty)$; 3) $(-\infty;\infty)$; 4) $(-\infty;-1)\cup(15;\infty)$; 5) $\varnothing$; 6) $(-\infty;\infty)$; 7) $(-\infty;0)\cup(4;\infty)$; 8) $(-2;2)$. **6.24.** 1) $(-3;-2)\cup(2;3)$; 2) $(-1;5)$; 3) $(1-\sqrt{3};2-\sqrt{2})$; 4) $(-1;2)\cup(3;6)$;

5) $(-\infty;-4)\cup(2;\infty)$. **6.25.** $\left(\frac{5}{3};\infty\right)$. **6.29.** 1) $\left(-\infty;\frac{3}{2}\right)\cup\left(\frac{7}{3};\infty\right)$; 2) $(-\infty;-3)\cup$ $\cup[2;3)$; 3) $(1;2)\cup(2;3)$; 4) $[-1;2]\cup\{3;4\}\cup[5;\infty)$; 5) $(-7;2)\cup[5;\infty)$; 6) $(-5;\ \infty)$; 7) $(-\infty;2)\cup(5;\infty)$; 8) $(-7;-3)$; 9) $(-\infty;-6)\cup[-2;0)\cup[3;\infty)$; 10) $\left(-\frac{9}{2};\ -2\right)\cup(3;\infty)$; 11) $(-7;-4)\cup(-4;1)$. **6.30.** 1) $\left(-\infty;\frac{3}{2}\right]\cup\left(\frac{7}{3};\infty\right)$; 2) $(-\infty;-5]\cup\cup(5;\infty)$; 3) $(-\infty;-3]\cup[1;\infty)$; 4) $(-\infty;-2]\cup[0;2]$; 5) $[-4;1]$ 6) $(-\infty;-4]\cup[5;6]$. **6.34.** 1) $(18;\infty)$; 2) $\left[-\frac{1}{3};\infty\right)$; 3) $[0,5;1)$; 4) $[1;\infty)$; 5) $[-1;0)\cup(3;4]$; 6) $[-1;1,7)$; 7) $\left(\frac{3}{4};2\right)$; 8) $\left[-4;\frac{3-\sqrt{2}}{2}\right)\cup(1;\infty)$; 9) $(-2;2)$; 10) $(-\infty;\infty)$; 11) $(5;\infty)$; 12) $(-\infty;1)$; 13) $(1;\infty)$; 14) $[-3;1)$.

7

7.20. a) $a_1=-7,\quad d=3$; b) $a_1=1.05;\quad d=0.8$; c) $a_1=0.25;\quad d=1.5$. **7.21.** $1,4,7$. **7.22.** $14,10,6,2$ or $2,6,10,14$. **7.23.** 0. **7.24.** 55. **7.25.** 610. **7.26.** 44. **7.27.** $n=7$. **7.28.** $5,7,9$. **7.29.** $\frac{6-\sqrt{6}}{6},1,\frac{6+\sqrt{6}}{6}$. **7.30.** 55. **7.31.** 1. **7.32.** $3,5,7$. **7.33.** $2,6,18$ or $2,-6,18$. **7.34.** $(3,-6,12,-24)$. **7.35.** $b_1=-7,\ n=10$. **7.36.** $b_1=5, b_5=406$. **7.37.** $n=5$. **7.38.** $\left(\frac{1}{5},1,5,25\right)$. **7.39**. $\frac{1}{625},\ 15625$. **7.40**. 1 082,43 UAH. **7.41.** $10,2,\frac{2}{5},...$ or $15,-3,\frac{3}{5},...$ **7.42.** $b_1=400,\ q=-\frac{2}{3}$. **7.43.** $S=3$. **7.44.** $6,2,\frac{2}{3},...$ **7.45.** $\frac{1}{16}$. **7.46.** $\left\{\frac{1}{3},\frac{2}{5}\right\}$ **7.47.** $\left\{-\frac{7}{9},\frac{1}{2}\right\}$. **7.48.** 931. **7.49.** $(3,18,33)$ or $(27,18,9)$. **7.50.** $(5,15,45)$. **7.51.** $\{3,6,12;18\}$. **7.52.** $\left\{\frac{10}{3},\frac{4}{3}\right\}$ or $\left\{-\frac{3}{4},-\frac{3}{10}\right\}$.

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Навчальний посібник для слухачів підготовчого відділення

(англ. мовою)

Самостійне електронне текстове мережеве видання

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Викладено всі теми шкільного курсу алгебри на основі освітньої програми середньої школи України. Докладно розглянуто методи розв'язання задач з усіх тем алгебри і наведено рішення типових прикладів. Запропоновано багато завдань для самостійного вирішення.

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