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HIGHER MATHEMATICS

Guidelines to independent work
on the topic "Series"

for Bachelor's (first) degree students
of subject area 12 "Information Technology"

Kharkiv
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The sufficient theoretical material on the academic discipline and typical examples are presented to help students master the material on the topic "Series" and apply the obtained knowledge to practice. A detailed description and guidelines for doing tasks for independent work, references, a list of theoretical questions and a test for self-assessment are given in order to improve students' knowledge of the topic. The professional competences that students acquire as a result of studying the theoretical material and completing practical tasks on this topic are defined.

For Bachelor's (first) degree students of subject area 12 "Information Technology".

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Introduction

Series is one of the basic concepts of mathematical analysis and mathematics in general. This branch of mathematics is of great practical importance and is widely used in various fields of human activity.

In the guidelines, the most principal topics of numerical series are stated in brief. The purpose of the guidelines is to consider the main definitions, concepts, properties, theorems and formulas of numerical series theory and to review basic methods and principles of testing numerical series for convergence: a necessary condition for convergence of a numerical series; sufficient conditions for convergence of numerical series with positive terms (d'Alembert's ratio test, Cauchy's radical test, Cauchy's integral test, comparison test); sufficient conditions for the convergence of alternating series. As a result of elaboration of the material presented in the guidelines, the student will be able to distinguish between the types of series and choose the methods for testing series for convergence, investigate the convergence of numerical series with positive and alternating terms, calculate the sum of a series.

These guidelines are intended for students and can be used for studying under the guidance of a lecturer as well as independently. This material is presented in the form of exercises and tasks for independent work, theoretical questions and tests for self-assessment.

1. The basic concepts of numerical series

This section is mostly devoted to notational issues as well as making sure we can do some basic manipulations with infinite series.

First, we should note that we will refer to infinite series as simply series. If we ever need to work with both infinite and finite series we'll be more careful with terminology, but in most sections we'll be dealing exclusively with infinite series and so we'll just call them series.

Let us start with a sequence $\{a_n\}_{n=1}^{\infty}$ (note here that $n=1$ is for convenience, it can be any other number) and define the following:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\dots$$

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots + a_n = \sum_{i=1}^{n} a_i$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is an **infinite sequence** of numbers $a_1, a_2, a_3 \dots$, that is, there is a law according to which any member of the sequence a_n can be found by its number n. In other words, $a_n = f(n)$ is a function of a natural argument: $n \in \mathbf{N} \to a_n \in \mathbf{R}$.

A **numerical series** is an infinite sum of members of a numerical sequence denoted as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots \qquad (1)$$

In expression (1) the notation a_n with any number n is called the ***n*-th term of series**; here $a_i, i \in \mathbf{N}$ are **terms of series** (or sometimes elements or members of a series).

For example, consider a numerical series:

$$\sum_{n=1}^{\infty} \frac{1}{5n+1},$$

where $a_n = \frac{1}{5n+1}$ is the ***n*-th term of the series**; $a_2 = \frac{1}{11}, a_7 = \frac{1}{36}$ are terms of the series.

The sum S_n of a finite number of the first n terms is called the ***n*-th partial sum of a series**:

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots + a_n = \sum_{i=1}^{n} a_i . \qquad (2)$$

Also recall that the symbol $\sum$ is used to represent this summation and it has a variety of names. The most common names are: **series notation**,

summation notation, and **sigma notation**. In $\sum_{i=1}^{n} a_i$ the i is called **the index of summation** or just **index** for short, and note that the letter we use to represent the index does not matter. So for example the following series are all the same. The only difference is the letter we've used for the index:

$$\sum_{i=0}^{\infty} \frac{10}{i^3+1} = \sum_{k=0}^{\infty} \frac{10}{k^3+1} = \sum_{n=0}^{\infty} \frac{10}{n^3+1}, \text{ etc.}$$

It is important to note again that the index can start at whatever value the sequence of series terms starts at.

A numeric series is called **convergent** if there is a finite limit

$$S = \lim_{n\to\infty} S_n \neq \infty$$

of its partial sums; then we write

$$\sum_{n=1}^{\infty} a_n = S \quad \text{or} \quad \sum_{n=1}^{\infty} a_n < \infty,$$

where the value S is called **a sum of the series.** The **remainder of the convergent series** $r_k = \sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \ldots$ tends to 0 while $n \to \infty$.

Otherwise, that is, if the limit of the sequence of its partial sums is equal to infinity or does not exist at all, the numerical series is called **divergent** and can be written as

$$\sum_{n=1}^{\infty} a_n = \infty.$$

While considering a numerical series, the following two problems are solved practically:

1) to investigate the series for convergence;

2) to find its sum if the series converges.

Example 1. Investigate the convergence of the series using the definition, and if it converges, find its sum:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Solution. In this example $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ is the n-th term of the given series. Then

$$S_n = a_1 + a_2 + a_3 + a_4 + ... + a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + ... + \frac{1}{n(n+1)} =$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + ... + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

It is obvious that the limit of sequence of partial sums of this series exists and it is equal to 1:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - \lim_{n\to\infty}\frac{1}{n+1} = 1, \quad S = \lim_{n\to\infty} S_n = 1 \neq \infty.$$

Therefore, the series converges and its sum is equal to 1.

Example 2. Investigate the convergence of the series using the definition, and if it converges, find its sum:

$$\sum_{n=2}^{\infty} \frac{24}{9n^2 - 12n - 5}.$$

Solution. Here

$$a_n = \frac{24}{9n^2 - 12n - 5}$$

is the n-th term. Let us expand the n-th term of this series into partial fractions:

$$a_n = \frac{24}{9n^2 - 12n - 5} = \frac{24}{(3n+1)(3n-5)} = \left|(3n+1) - (3n-5) = 6\right| =$$

$$= 4\frac{(3n+1) - (3n-5)}{(3n+1)(3n-5)} = \frac{4}{3n-5} - \frac{4}{3n+1}.$$

Thus

$$a_n = 4\left(\frac{1}{3n-5} - \frac{1}{3n+1}\right).$$

Let us find the partial sum:

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + ... + a_n = 4\left(1 - \frac{1}{7} + \frac{1}{4} - \frac{1}{10} + \frac{1}{7} - \frac{1}{13} + \right.$$

$$\left. + ... + \frac{1}{3n-11} - \frac{1}{3n-5} + \frac{1}{3n-8} - \frac{1}{3n-2} + \frac{1}{3n-5} - \frac{1}{3n+1}\right).$$

Then the limit of sequence of partial sums of this series exists, and it is equal to 5:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 4\left(1 + \frac{1}{4}\right) = 5, \quad S = \lim_{n\to\infty} S_n = 5 \neq \infty.$$

Therefore, the series converges, and its sum is equal to 5.

Example 3. Investigate the convergence of the series using the definition, and if it converges, find its sum:

$$\sum_{n=1}^{\infty} aq^{n-1} \ (a \neq 0, q \in \mathbf{R}).$$

Solution. It is a numerical series of the geometric progression with the denominator q:

$$\sum_{n=1}^{\infty} aq^{n-1} = a + aq + aq^2 + ... + aq^9 + ... \ (a \neq 0, q \in \mathbf{R}) \qquad (3)$$

As you know, for this series, the partial sums have the form:

$$S_n = a\frac{1-q^n}{1-q}.$$

When we find a limit at $n \to \infty$, this leads to two different situations:

- if $|q| < 1$, series (3) converges $\sum_{n=1}^{\infty} aq^{n-1} = \frac{a}{1-q}$; (4)
- if $|q| \geq 1$, series (3) diverges.

Note here that the series of geometric progression is one of the important (special) series.

Example 4. Investigate the convergence of the series using the definition, and if it converges, find its sum:

$$\sum_{n=0}^{\infty} (-1)^n .$$

Solution. The series $\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+...$ diverges. In this case we really don't need a general formula for the partial sums to determine the convergence of this series. Let's just write down the first few partial sums:

$$S_1 = 1-1 = 0,$$
$$S_2 = 1-1+1 = 1,$$
$$S_3 = 1-1+1-1 = 0,$$
$$S_4 = 1-1+1-1+1 = 1,$$

...

...

$$S_{2n} = 1,$$
$$S_{2n+1} = 0,$$

..., etc.,

so, it looks like the sequence of partial sums is

$$\{S_n\}_{n=0}^{\infty} = \{1,0,1,0,1,0,1,0,...\},$$

and this sequence diverges since $\lim_{n\to\infty} S_n$ does not exist. Therefore, the series also diverges.

Example 5. Investigate the convergence of the series $1+1+1+1+1+1+1+...$ using the definition, and if it converges, find its sum.

Solution. This series diverges, as $S_n = n$ and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$.

We'll leave this section with an important warning about terminology. Do not get sequences and series confused! A sequence is a list of numbers written in a specific order while an infinite series is a limit of a sequence of finite series and hence, if it exists it will be a single value.

2. The properties of numerical series

We'll start off with basic arithmetic with infinite series as we'll need to be able to do that on occasion. We have the following properties.

1. The series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} \lambda a_n \quad (\lambda \in R, \quad \lambda \neq 0)$$

either simultaneously converge or simultaneously diverge. And if $\sum_{n=1}^{\infty} a_n = S$, $\sum_{n=1}^{\infty} \lambda a_n = \lambda S$.

2. If

$$\sum_{n=1}^{\infty} a_n = S_1, \qquad \sum_{n=1}^{\infty} b_n = S_2$$

are both convergent series,

$$\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n \pm b_n)$$

is also convergent and the sum is equal to $S_1 \pm S_2$.

3. The series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=k+1}^{\infty} a_n \quad (k > 1)$$

either simultaneously converge or simultaneously diverge, that is, rejecting or joining a finite number of members of the series does not affect its convergence.

The first property is simply telling us that we can always factor a multiplicative constant out of an infinite series. Recall that an initial value of the index of the series can start at any value.

The second property says that if we add/subtract series, all we really need to do is add/subtract the series terms. Note as well that in order to add/subtract series we need to make sure that both have the same initial value of the index and the new series will also start at this value.

Let's briefly discuss multiplication of series. We'll start with

$$\left(\sum_{n=0}^{\infty} a_n\right)\cdot\left(\sum_{n=0}^{\infty} b_n\right) \neq \sum_{n=0}^{\infty} (a_n \cdot b_n).$$

To make sure that this is true, consider the following product of two finite sums:

$$(2+x)(3-5x+x^2) = 6-7x-3x^2+x^3.$$

It was just the multiplication of two polynomials. Each is a finite sum and so it makes the point. In doing the multiplication we do not just first multiply the constant terms, then the x terms, and so on. Instead we have to multiply 2 by each team of the second polynomial, then multiply x by each team of the second polynomial and finally combine the similar terms.

Multiplying infinite series needs to be done in the same manner. Multiplication implies the following:

$$\left(\sum_{n=0}^{\infty} a_n\right)\cdot\left(\sum_{n=0}^{\infty} b_n\right) = (a_0+a_1+a_2+a_3+a_4+...)\cdot(b_0+b_1+b_2+b_3+b_4+...).$$

To do this multiplication we would have to distribute first a_0, then a_1 through the second term, etc., then combine the similar terms. This is pretty much impossible since both series have an infinite set of terms in them, however the following formula can be used to determine the product of two series:

$$\left(\sum_{n=0}^{\infty} a_n\right)\cdot\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n,$$

where $c_n = \sum_{i=0}^{n} a_i \cdot b_{n-i}$.

Neither can we say much about the convergence of the product. Even if both of the original series are convergent, it is possible for the product to be divergent. The reality is that multiplication of series is a somewhat difficult process and in general is avoided if possible.

3. The strategy for series

Note that here we give a general set of guidelines for examining series for convergence. For some series, more than one test can be applied. That is why you should always go completely through the guidelines and identify all possible tests that can be used for a given series. Once this has been done, you can identify the test that you feel will be the easiest for you to use.

3.1. The necessary condition for a series convergence

So, we have considered several series in the first section. Recall here that two of the series (examples 1 and 2) converged, two diverged (examples 4 and 5) and the convergence or divergence of one series depended on the parameter (example 3). Let us go back and examine the series terms for each of these. For each of the series let's take the limit as n going to infinity of the series general terms (not the partial sums!).

- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n(n+1)} = 0$, this series converged.
- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{24}{9n^2 - 12n - 5} = 0$, this series converged.

About special series:

$$\sum_{n=1}^{\infty} aq^{n-1} = a + aq + aq^2 + ... + aq^9 + ... \ (a \neq 0, q \in \mathbf{R}),$$

where

- $a_n = aq^{n-1}$, $a \neq 0, q \in \mathbf{R}$ as we have known that

if $|q| < 1$, the series converges, and $\sum\limits_{n=1}^{\infty} aq^{n-1} = \frac{a}{1-q}$;

if $|q| \geq 1$, this series diverges.

- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)^n$ does not exist, this series diverged.

And, finally,

- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 1 = 1$, this series diverged.

Notice that for the two series that converged the series term itself was zero in the limit. This will always be true for convergent series and leads to the following theorem.

Theorem. If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0. \tag{5}$$

Corollary. If condition (5) is not satisfied, the series diverges, namely: if $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ diverges (in literature this condition is sometimes called the divergence test). Otherwise, if $\lim_{n\to\infty} a_n = 0$, the series can either converge or diverge.

It is the corollary of the theorem that is usually used in the practical study of series for convergence.

3.2. Sufficient conditions for convergence of series with positive terms

Let's consider a general set of guidelines to help us determine the convergence of the series $\sum_{n=1}^{\infty} a_n$ with positive: $\forall n \in \mathbf{N}\ a_n > 0$. But note here that, due to the first property of series, they can also be applied to series with negative terms.

Theorem. *(The ratio test or d'Alembert's test.)* Let for the series $\sum_{n=1}^{\infty} a_n$ ($a_n > 0$) there is a limit

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}. \tag{6}$$

Then,

- if $l < 1$, the series converges;
- if $l > 1$, the series diverges. (7)

It is important to remember: at $l = 1$ d'Alembert's test is inapplicable. In this case it is necessary to apply some other test.

It is recommended to use the d'Alembert's test if there are exponential or/and factorial functions in the formula of the general term.

Theorem. *(The Cauchy radical test.)* Assume that for a series with nonnegative terms $\sum_{n=1}^{\infty} a_n$ $(a_n > 0)$ there exists

$$l = \lim_{n\to\infty} \sqrt[n]{a_n}. \tag{8}$$

Then,

- if $l < 1$, the series converges;
- if $l > 1$, the series diverges. (9)

It is important to remember: at $l = 1$ the Cauchy test is inapplicable; it is necessary to apply some other test.

It is recommended to use the Cauchy radical test if there is a degree n in the formula of a general term of a series.

Theorem. *(The Cauchy integral test.)* Let's assume we have $\sum_{n=1}^{\infty} a_n$. Suppose that $f(x)$ is a positive decreasing function on the interval $x \in [1;+\infty)$ and that $a_n = f(n)$.

Then,

- if $\int_1^{\infty} f(x)dx$ is convergent, the series $\sum_{n=1}^{\infty} a_n$ also converges;
- if $\int_1^{\infty} f(x)dx$ is divergent, the series $\sum_{n=1}^{\infty} a_n$ also diverges.

That is, the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)dx$ are simultaneously either convergent or divergent.

Example 6. Using the Cauchy integral test, investigate the convergence of the harmonic series (or Dirichlet series):

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad (p > 0). \tag{10}$$

Solution.

Let us consider the corresponding integral:

$$\int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} x^{-p} dx = \lim_{b\to\infty} \int_1^b x^{-p} dx = \lim_{b\to\infty} \left(\frac{1}{1-p} x^{1-p} \right)\Bigg|_1^b =$$

$$= \lim_{b\to\infty} \frac{1}{1-p} \left(b^{1-p} - 1 \right) = \left[\begin{array}{ll} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{array} \right. .$$

In case $p = 1$, we have

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln b = \infty .$$

Thus, as a result of the study of the convergence of the Dirichlet series, we obtained the following result:

- if $p > 1$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges;
- if $p \le 1$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ diverges. (11)

Theorem. *(The comparison test.)* Suppose that we have two series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$. Let the terms of this series be such that $0 < a_n \le b_n$.

Then,

- if $\sum\limits_{n=1}^{\infty} b_n$ is convergent, the series $\sum\limits_{n=1}^{\infty} a_n$ also converges;
- if $\sum\limits_{n=1}^{\infty} a_n$ is divergent, the series $\sum\limits_{n=1}^{\infty} b_n$ also diverges.

In other words, from the convergence of the series $\sum_{n=1}^{\infty} b_n$ the convergence of the series $\sum_{n=1}^{\infty} a_n$ follows, and vice versa, from the divergence of the series $\sum_{n=1}^{\infty} a_n$ the divergence of the series $\sum_{n=1}^{\infty} b_n$ follows.

Theorem. *(The limit comparison test.)* Let the terms of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be nonnegative and there exists $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, $l \neq 0$, $l \neq \infty$, then both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge simultaneously.

In practice we use as **comparison standards (or special series)**, for example, series (3) and (10), whose conditions of convergence are known (see (4) and (11) accordingly).

3.3. Alternating series. Absolute and conditional convergence

An **alternating series** is a series with the terms which are real numbers of any sign.

For example,

$$\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} - \frac{1}{7^2} + \ldots + (-1)^{\frac{n(n-1)}{2}} \frac{1}{n^2} + \ldots$$

An alternating series is called **alternating in sign** if any of its two neighbouring terms have different signs, i.e. series like

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad (a_n > 0). \qquad (12)$$

An alternating series is called **absolutely converging** if the series $\sum_{n=1}^{\infty} |a_n|$ converges, namely: $\sum_{n=1}^{\infty} |a_n| < \infty$.

An alternating series is called **conditionally converging** if the series $\sum_{n=1}^{\infty} a_n$ converges and the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, the alternating series $\sum_{n=1}^{\infty} a_n$ also converges.

Theorem. *(Leibniz criterion.)* If the elements $\{a_n\}_{n=1}^{\infty}$ of an alternating series (12) form a monotonically decreasing sequence which tends to zero, i.e. if:

1) $\lim_{n\to\infty} a_n = 0$;

2) $a_n > a_{n+1}, \forall n$,

the series converges. Moreover, its sum is positive and less than the first term of the series: $0 < S < a_1$.

Series for which the Leibniz criterion is satisfied are called **Leibniz-type series**.

Corollary. For Leibniz-type series the absolute value of the remainder r_n is less than the absolute value of its first term:

$$|S - S_n| = |r_n| \leq a_{n+1}, \quad (13)$$

where

$$r_n = (-1)^{n+2} a_{n+1} + (-1)^{n+3} a_{n+2} + \ldots$$

So, with that said here is a *set of guidelines for determining the convergence of a series*.

1. With a quick glance, does it look like the series terms don't converge to zero in the limit, i.e. does

$$\lim_{n\to\infty} a_n \neq 0?$$

If so, use the divergence test. Note that you should only do the divergence test if a quick glance suggests that the series terms may not converge to zero in the limit.

2. Is the series a p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad (p > 0)$$

or a geometric series

$$\sum_{n=0}^{\infty} aq^n \text{ or } \sum_{n=1}^{\infty} aq^{n-1} \ (a \neq 0, q \in \mathbf{R})?$$

If so, use the fact that p-series will only converge if $p > 1$ and geometric series will only converge if $|q| < 1$. Remember as well that often some algebraic manipulation is required to get a geometric series into the correct form.

3. Is the series similar to a p-series or a geometric series? If so, try the comparison test.

4. Is the series a rational expression involving only polynomials or polynomials under radicals (i.e. a fraction involving only polynomials or polynomials under radicals)? If so, try the comparison test and/or the limit comparison test. Remember however, that in order to use the comparison test and the limit comparison test the series terms all need to be positive.

5. Does the series contain factorials or constants raised to powers involving n? If so, the ratio test may work. Note that if the series term contains a factorial, the only test that we've got that will work is the ratio test.

6. Can the series terms be written in the form

$$(-1)^n a_n \quad \text{or} \quad (-1)^{n-1} a_n \quad (a_n > 0)?$$

If so, the alternating series test may work.

7. Can the series terms be written in the form

$$a_n = (b_n)^n?$$

If so, the root test may work.

8. If $a_n = f(n)$ for some positive decreasing function and $\int_c^{\infty} f(x)\,dx$ is easy to evaluate, the integral test may work.

Again, remember that these are only a set of guidelines and not a set of hard and fast rules to use when trying to determine the best test to use on a series. If more than one test can be used, try to use the test that will be the easiest for you to use and remember that what is easy for someone else may not be easy for you!

Theoretical questions for self-assessment

1. Expand the content of the concept of numerical series.
2. Describe the properties of numerical series.
3. What is the necessary condition for convergence of series?
4. How do you understand the phrase "series with positive terms"?
5. What sufficient conditions for convergence of series with positive terms do you know?
6. Give some examples of special series. Why are they called special? How are they used?
7. What is the essence of the comparison test?
8. Formulate d'Alembert's test, Cauchy radical test, integral test for convergence of series.
9. Give examples of general terms of a series matching each sufficient test.
10. Explain the concept of alternating series.
11. What is the content of the Leibniz criterion?
12. Give a definition of absolute and conditional convergence of series.
13. Give examples of absolutely converging series.
14. Give examples of conditionally converging series.

Test for self-assessment

1. What is the twenty-first term of the sequence given by $x_n = 4n - 3$:

a) 6; b) 72; c) 81; d) 87?

2. If n is a natural number, what is the sixth term of the sequence given by $x_n = 4 \cdot (0,5)^n$:

a) 0.0625; b) 0.125; c) 1; d) 64?

3. What is the fifth term of the sequence $a_n = \left\{ \left(\frac{1}{n} \right)^{n-1} \right\}$:

a) $\frac{1}{5}$; b) $\frac{1}{25}$; c) $\frac{1}{125}$; d) $\frac{1}{625}$?

4. For the sequence defined by $a_n = n^2 - 5n + 2$, what is the smallest value of n for which a_n is positive:

a) 3; b) 4; c) 5; d) 6?

5. The n-th hexagonal number is given by $h_n = n(2n-1)$. What is the sixth hexagonal number:

a) 17; b) 22; c) 66; d) 150?

6. The n-th term of a sequence is given by $x_n = 3n^2 - 1$. Which term of the sequence is equal to 866:

a) the fifteenth term; b) the sixteenth term;
c) the seventeenth term. d) the twentieth term?

7. The n-th term of a sequence is given by $x_n = \dfrac{n^2 - 1}{n+5}$. Which term of the sequence is equal to 4:

a) the third term; b) the seventh term;
c) the eighth term; d) the tenth term?

8. What is the partial sum of the squares of the first eight natural numbers:

a) 64; b) 140; c) 204; d) 1296?

9. The partial sums of the first n and $n+1$ numbers of the Fibonacci sequence are both divisible by 11. What is the smallest value of n for which this is true:

a) $n = 11$; b) $n = 10$; c) $n = 9$; d) $n = 8$?

10. What is the value of $\sum_{k=1}^{4} 5k^2$:

a) 150; b) 250; c) 500; d) 750?

11. What is the value of $\sum_{k=1}^{10} k(k-1)(k+1)$:

a) 2640; b) 2970; c) 3025; d) 3080?

12. What is the value of $\sum_{k=1}^{12} k^2(k-3)$:

a) 234; b) 4134; c) 5109; d) 8034?

13. What is the sum of the whole numbers from 10 to 100 inclusive:

a) 4 939; b) 4 995; c) 5 005; d) 5 050?

14. What is the sum of the whole numbers from 12 to 20 inclusive:

a) 39 744; b) 38 016; c) 37 566; d) 35 388?

15. The partial sum of the first n natural numbers is given by the formula: $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. If $\sum_{k=1}^{n} k = 496$, what is the value of n:

a) $n=31$; b) $n=32$; c) $n=33$; d) $n=12325$?

16. The partial sum of the first n natural numbers is given by the formula: $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$. If $\sum_{k=1}^{n} k^2 = 130n$, what is the value of n:

a) $n=13$; b) $n=18$; c) $n=19$; d) $n=26$?

17. Evaluate $\sum_{i=1}^{100} i = 1+2+3+...+98+99+100$:

a) $5000 = 50 \cdot 100$; b) $4950 = \frac{99 \cdot 100}{2}$;

c) $5050 = \frac{100 \cdot 101}{2}$; d) $2500 = 50 \cdot 50$.

18. Let $S_n = \sum_{k=1}^{n} (-1)^{k^2+k}$, then $S_n =$:

a) undefined; b) n;

c) 1; d) 0 or 1 depending on the value of n?

19. Let $S_n = \sum_{k=1}^{n} k^2$. Which of the following is not equal to S_6 :

a) $\frac{6 \cdot 7 \cdot 14}{6}$; b) $S_7 - 7^2$;

c) $S_5 + 6^2$; d) 91?

20. If the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n}+2}$ converges, find a_1^2, and if it diverges, find a_4^2:

a) 1; b) $\frac{1}{3}$;

c) $\frac{1}{9}$; d) nothing can be said about this series.

21. If $\sum_{n=1}^{\infty} arcctg\frac{5}{4+n}$ converges, find its first term:

a) 1; b) π;

c) 0.25π; d) diverges.

22. If $\sum_{n=1}^{\infty}\left(\frac{2n-3}{5n-4}\right)^{2n}$ diverges, find its first term, but if the given series converges, find its second term:

a) 1; b) $\frac{1}{1296}$;

c) $\frac{1}{36}$; d) $\frac{1}{99}$.

23. For which series is the necessary condition for convergence not fulfilled:

a) $\sum_{n=1}^{\infty}\frac{3n+1}{4n+5}$; b) $\sum_{n=1}^{\infty}\frac{2}{11n+2}$;

c) $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n(n+1)}}$; d) $\sum_{n=1}^{\infty}\frac{2n^2}{(3n-1)^2}$?

24. For which series is the necessary condition for convergence satisfied and in order to establish the convergence or divergence of the series, what additional research is needed:

a) $\sum_{n=1}^{\infty}\frac{\sqrt[3]{n}}{\sqrt{n}(n+2)}$; b) $\sum_{n=1}^{\infty}\frac{3n-1}{7n}$;

c) $\sum_{n=1}^{\infty} n^3\sin\frac{1}{n^3}$; d) $\sum_{n=1}^{\infty}\left(\frac{n+3}{n+1}\right)^{n}$?

Correct answers with explanations for the test

1 c

Solution. To find the twenty-first term, replace *n* by 21: therefore $x_{21} = 4 \times 21 - 3 = 84 - 3 = 81$.

2 a

Solution. To find the sixth term, replace n by 6: therefore $x_6 = 4 \cdot (0.5)^6 = 4 \cdot 0.015625 = 0.0625$.

3 d

Solution. To find the fifth term, replace n with 5. Therefore $\{a_5\} = \left\{\left(\frac{1}{5}\right)^4\right\} = \frac{1}{625}$.

4 c

Solution.

$a_1 = 1^2 - 5 \times 1 + 2 = 1 - 5 + 2 = -2 < 0;$

$a_2 = 2^2 - 5 \times 2 + 2 = 4 - 10 + 2 = -4 < 0;$

$a_3 = 3^2 - 5 \times 3 + 2 = 9 - 15 + 2 = -4 < 0;$

$a_4 = 4^2 - 5 \times 4 + 2 = 16 - 20 + 2 = -2 < 0;$

$a_5 = 5^2 - 5 \times 5 + 2 = 25 - 25 + 2 = 2 > 0.$

Therefore the smallest value of n for which a_n is positive is $n = 5$.

5 c

Solution. To find the sixth hexagonal number, replace n with 6. Therefore $h_6 = 6(2 \times 6 - 1) = 6(12 - 1) = 6 \times 11 = 66$.

6 c

Solution. If $x_n = 866$,

$\Rightarrow 3n^2 - 1 = 866;$

$\Rightarrow 3n^2 = 866 + 1 = 867;$

$\Rightarrow n^2 = 867 : 3 = 289;$

$\Rightarrow n = \sqrt{289} = 17.$

Therefore the seventeenth term is equal to 866.

7 b

Solution. If $x_n = 4$,

$\Rightarrow \frac{n^2 - 1}{n + 5} = 4;$

$\Rightarrow n^2 - 1 = 4(n + 5);$

$\Rightarrow n^2 - 1 = 4n + 20;$

$\Rightarrow n^2 - 4n - 21 = 0.$

That quadratic equation can be factored into: $(n-7)(n+3) = 0$, which has solutions $n = 7$, or $n = -3 \Rightarrow n = 7$ since n cannot be negative.

So, the seventh term is equal to 4.

8 c

Solution.

$1^2 = 1;$

$2^2 = 4;$

$3^2 = 9;$

$4^2 = 16;$

$5^2 = 25;$

$6^2 = 36;$

$7^2 = 49;$

$8^2 = 64.$

Therefore the partial sum of squares of the first eight natural numbers is $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 = 204$.

9 b

Solution. These are the first few terms of the Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,...$$

The partial sum of the first term $= 0$.

The partial sum of the first two terms $= 0 + 1 = 1$.

The partial sum of the first three terms $= 0 + 1 + 1 = 2$.

The partial sum of the first four terms $= 0 + 1 + 1 + 2 = 4$.

The partial sum of the first five terms $= 0 + 1 + 1 + 2 + 3 = 7$.

The partial sum of the first six term $= 0 + 1 + 1 + 2 + 3 + 5 = 12$.

The partial sum of the first seven terms $= 0 + 1 + 1 + 2 + 3 + 5 + 8 =$ $= 20$.

The partial sum of the first eight terms $= 0 + 1 + 1 + 2 + 3 + 5 + 8 +$ $+ 13 = 33$.

The partial sum of the first nine terms $= 0 + 1 + 1 + 2 + 3 + 5 + 8 +$ $+ 13 + 21 = 54$.

The partial sum of the first ten terms $= 0 + 1 + 1 + 2 + 3 + 5 + 8 +$ $+ 13 + 21 + 34 = 88$.
The partial sum of the first eleven terms $= 0 + 1 + 1 + 2 + 3 + 5 +$ $+ 8 + 13 + 21 + 34 + 55 = 143$.
33 is divisible by 11, but 54 is not. Then 88 and 143 are both divisible by 11. So, $n=10$ is the smallest value of n for which the sums of the first n and $n+1$ numbers are both divisible by 11.

10 a

Solution. Use the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} .$$

Therefore $\sum_{k=1}^{4} 5k^2 = 5\sum_{k=1}^{4} k^2 = 5\cdot\frac{1}{6}\cdot 4\cdot 5\cdot 9 = 5\cdot 30 = 150$.

11 b

Solution.

$$\sum_{k=1}^{10} k(k-1)(k+1) = \sum_{k=1}^{10} k(k^2-1) = \sum_{k=1}^{10} k^3 - k = \sum_{k=1}^{10} k^3 - \sum_{k=1}^{10} k .$$

Use the formulae $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ and $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Therefore $\sum_{k=1}^{10} k^3 - \sum_{k=1}^{10} k = \left(\frac{10\cdot 11}{2}\right)^2 + \frac{1}{2}\cdot 10\cdot 11 = 3025 - 55 = 2970$.

12 b

Solution.

$$\sum_{k=1}^{12} k^2(k-3) = \sum_{k=1}^{12} k^3 - 3k^2 = \sum_{k=1}^{12} k^3 - 3\sum_{k=1}^{12} k^2 .$$

Use the formulae $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ and $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

Then $\sum_{k=1}^{12} k^3 - 3\sum_{k=1}^{12} k^2 = \left(\frac{12\cdot 13}{2}\right)^2 - 3\cdot\frac{1}{6}\cdot 12\cdot 13\cdot 25 = 6084 - 1950 = 4134$.

13 c

Solution. To find the sum from 10 to 100 inclusive, we find the difference between the sum of the whole numbers from 1 to 100 and the sum of the whole numbers from 1 to 9. (This is 9, not 10, because 10 must be included in the answer). Use the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Then $\sum_{k=10}^{100} k = \sum_{k=1}^{100} k - \sum_{k=1}^{9} k = \frac{100 \cdot 101}{2} - \frac{9 \cdot 10}{2} = 5050 - 45 = 5005.$

14 a

Solution. To find the sum from 12 to 20 inclusive, we find the difference between the sum of the cube numbers from 1 to 20 and the sum of the cube numbers from 1 to 11. (This is 11, not 12, because 12^3 must be included in the answer).

Use the formula

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Then

$$\sum_{k=12}^{20} k^3 = \sum_{k=12}^{20} k^3 - \sum_{k=1}^{11} k^3 = \left(\frac{20 \cdot 21}{2}\right)^2 - \left(\frac{11 \cdot 12}{2}\right)^2 = 44100 - 4356 = 39744.$$

15 a

Solution.

$$\sum_{k=1}^{n} k = \frac{1}{2} n(n+1) \text{ and } \sum_{k=1}^{n} k = 496.$$

$\Rightarrow \frac{1}{2} n(n+1) = 496;$

$\Rightarrow n(n+1) = 992;$

$\Rightarrow n^2 + n = 992;$

$\Rightarrow n^2 + n - 992 = 0.$

That quadratic equation can be factored into: $\Rightarrow (n-31)(n+32) = 0;$

$\Rightarrow n = 31$ since n must be a positive number.

16 c

Solution.

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \text{ and } \sum_{k=1}^{n} k^2 = 130n.$$

$\Rightarrow \frac{1}{6}n(n+1)(2n+1) = 130n;$

$\Rightarrow \frac{1}{6}n(n+1)(2n+1) = 130n$ since $n \neq 0$;

$\Rightarrow (n+1)(2n+1) = 780;$

$\Rightarrow 2n^2 + 3n + 1 = 780;$

$\Rightarrow 2n^2 + 41n - 38n - 779 = 0;$

$\Rightarrow n(2n+41) - 19(2n+41) = 0;$

$\Rightarrow (n-19)(2n+41) = 0;$

$\Rightarrow n = 19$ since n must be a whole number.

17 c

Solution. Using the formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

with $n = 100$, we have

$$\sum_{i=1}^{100} i = \frac{100 \cdot 101}{2} = 5050.$$

18 b

Solution. Note that for any natural number k, the exponent $k^2 + k = k(k+1)$ is an even number since it is the product of an even number and an odd number. Thus $(-1)^{k^2+k} = 1$, so

$$\sum_{k=1}^{n} (-1)^{k^2+k} = \sum_{k=1}^{n} 1 = \underbrace{1+1+...+1+1}_{n \text{ summands of } 1} = n.$$

19 a

Solution. The general formula for the sum of the squares of the first n positive integers is

$$1^2+2^2+3^2+...+n^2=\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}.$$

This problem asks about S_6, which is this sum with $n=6$. Substituting this value of n into the formula gives

$$S_6=1^2+2^2+3^2+4^2+5^2+6^2=\sum_{i=1}^{6} i^2=\frac{6\cdot 7\cdot 13}{6}.$$

This problem asks which answer choice is not equal to S_6. One of the choices for the answer is

$$\frac{6\cdot 7\cdot 14}{6}.$$

Comparing this to the value of S_6 we found, we see that

$$S_6\neq\frac{6\cdot 7\cdot 14}{6},$$

so, $\frac{6\cdot 7\cdot 14}{6}$ is the correct answer.

20 a

Solution. Let us compare this series with the Dirichlet series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

As $\frac{n}{\sqrt{n}+2}>\frac{1}{\sqrt{n}}$, and the Dirichlet series diverges, this series also diverges.

Then, $a_4^2=\frac{16}{16}=1.$

21 d

Solution. As $a_n=arcctg\,\frac{5}{4+n}$, let us check the necessary condition (5) for the convergence of the series. For this purpose we calculate the limit:

$$\lim_{n\to\infty} arcctg\,\frac{5}{4+n}=\lim_{n\to\infty} arcctg\,0=\frac{\pi}{2}\neq 0.$$

Since the limit of the general term of the series is not equal to zero ($\lim\limits_{n\to\infty} a_n \neq 0$), the necessary condition (5) of convergence is not fulfilled, then the given series $\sum\limits_{n=1}^{\infty} arcctg \frac{5}{4+n}$ diverges.

22 b

Solution. A general term of this series has n-th degrees, therefore let us use the Cauchy radical test. To do this, we calculate the limit (8):

$$l = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n-3}{5n-4}\right)^{2n}} = \lim_{n\to\infty}\left(\frac{2n-3}{5n-4}\right)^2 = \lim_{n\to\infty} \frac{2}{5}^2 < 1.$$

Then the given series $\sum\limits_{n=1}^{\infty}\left(\frac{2n-3}{5n-4}\right)^{2n}$ converges. So, $a_2 = \left(\frac{4-3}{10-4}\right)^4 = \frac{1}{1296}$.

23 a, c, d

Solution. The following corollary of the theorem is usually used in the practical test of series for convergence: if $\lim\limits_{n\to\infty} a_n \neq 0$, the series $\sum\limits_{n=1}^{\infty} a_n$ diverges. Otherwise, if $\lim\limits_{n\to\infty} a_n = 0$, the series can either converge or diverge.

The necessary condition for a series convergence is not fulfilled for the series:

a) $\sum\limits_{n=1}^{\infty} \frac{3n+1}{4n+5}$ as $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{3n}{4n} = \frac{3}{4} \neq 0$;

c) $\sum\limits_{n=1}^{\infty} \frac{n}{\sqrt{n(n+1)}}$ since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2}} = 1 \neq 0$;

d) $\sum\limits_{n=1}^{\infty} \frac{2n^2}{(3n-1)^2}$ because $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{2n^2}{3n^2} = \frac{2}{3} \neq 0$.

24 a

Solution. The following corollary of the theorem is usually used in the practical test of series for convergence: if $\lim\limits_{n\to\infty} a_n \neq 0$, the series $\sum\limits_{n=1}^{\infty} a_n$ diverges. Otherwise, if $\lim\limits_{n\to\infty} a_n = 0$, the series can either converge or diverge.

The necessary condition for a series convergence is satisfied for

$$\text{a) } \sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{\sqrt{n}(n+2)} \text{ as } \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^{\frac{1}{3}}}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{7}{2}}} = 0.$$

In order to establish the convergence or divergence of the series, additional research is needed. Let us consider the variants:

$$\text{b) } \sum_{n=1}^{\infty} \frac{3n-1}{7n}: \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{7n} = \frac{3}{7} \neq 0;$$

$$\text{c) } \sum_{n=1}^{\infty} n^3 \sin\frac{1}{n^3}: \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^3 \frac{1}{n^3} = 1 \neq 0;$$

$$\text{d) } \sum_{n=1}^{\infty} \left(\frac{n+3}{n+1}\right)^n: \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+3}{n+1}\right)^n = \left|1^{\infty}\right| =$$

$$= \lim_{n\to\infty} \left(1 + \frac{2}{n+1}\right)^n = \lim_{n\to\infty} \left[\left(1 + \frac{2}{n+1}\right)^{\frac{n+1}{2}}\right]^{\frac{2n}{n+1}} = \lim_{n\to\infty} e^{\frac{2n}{n+1}} = e^2 \neq 0.$$

Solution of typical example tasks of a test paper

In this section we consider how to apply different techniques of investigation for numerical series, which will help in solving typical tasks of an independent test paper.

Task 1. Test the numerical series $\sum\limits_{n=1}^{\infty} a_n$ for convergence, if the general term a_n of this series has the form:

a) $a_n = arcctg\dfrac{n^3+33}{n^2+22}$; b) $a_n = \dfrac{n^2+1}{(2n-1)!}$;

c) $a_n = \left(\dfrac{n+3}{n+1}\right)^{n^2}$; d) $a_n = \dfrac{n\sqrt{n}}{50n^3 - 12n + 1}$;

e) $a_n = \dfrac{1}{(n+1)\ln(n+1)}$.

Solution.

a) $\sum\limits_{n=1}^{\infty} arcctg\,\dfrac{n^3+33}{n^2+22}$.

Let us check the necessary condition (5) for the convergence of the series. For this purpose we calculate the limit:

$$\lim_{n\to\infty} arcctg\,\frac{n^3+33}{n^2+22} = \lim_{n\to\infty} arcctg\,\frac{n^3}{n^2} = \lim_{n\to\infty} arcctg\,n = \frac{\pi}{2} \neq 0.$$

Since the limit of the general term of the series is not equal to zero ($\lim\limits_{n\to\infty} a_n \neq 0$), the necessary condition (5) of convergence is not fulfilled; then the given series $\sum\limits_{n=1}^{\infty} arcctg\,\dfrac{n^2+3}{n+2}$ diverges.

b) $\sum\limits_{n=1}^{\infty} \dfrac{n^2+1}{(2n-1)!}$.

As the formula of the general term contains a factorial, we use d'Alembert test. For this purpose we calculate the limit (6):

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2+1}{(2(n+1)-1)!} \cdot \frac{(2n-1)!}{n^2+1} = \left|\frac{\infty}{\infty}\right| =$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \lim_{n\to\infty} \frac{(2n-1)!}{(2n-1)! \cdot 2n \cdot (2n+1)} = 1 \cdot \lim_{n\to\infty} \frac{1}{4n^2+2n} = \left|\frac{1}{\infty}\right| = 0 < 1.$$

Since $l < 1$, we conclude that the given series converges.

c) $\sum\limits_{n=1}^{\infty} \left(\dfrac{n+3}{n+1}\right)^{n^2}$.

The general term of this series has an n-th degree, therefore let us use the Cauchy radical test. To do this, we calculate the limit (8):

$$l=\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n+3}{n+1}\right)^{n^2}}=\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^{n}=\left|1^{\infty}\right|=$$

$$=\lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{n}=\lim_{n\to\infty}\left[\left(1+\frac{2}{n+1}\right)^{\frac{n+1}{2}}\right]^{\frac{2n}{n+1}}=\lim_{n\to\infty}e^{\frac{2n}{n+1}}=e^{2}>1.$$

As $l>1$, the given series diverges.

d) $\sum\limits_{n=1}^{\infty}\dfrac{n\sqrt{n}}{50n^3-12n+1}$.

We apply the limiting form of the comparison test. Firstly, find an appropriate series for comparison. For this purpose it is necessary to preserve senior degrees without numerical coefficients in the formula of the general term:

$$a_n=\frac{n\sqrt{n}}{50n^3-12n+1}\sim\frac{n\sqrt{n}}{n^3}=\frac{1}{n^{3/2}}=b_n.$$

As a series for comparison, choose the Dirichlet series $\sum\limits_{n=1}^{\infty}b_n=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{3/2}}$,

which according to (11) converges, as $p=\dfrac{3}{2}>1$.

Let us calculate the limit:

$$l=\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n\sqrt{n}}{50n^3-12n+1}\cdot\frac{n^{3/2}}{1}=$$

$$=\lim_{n\to\infty}\frac{n^3}{50n^3-12n+1}=\left|\frac{\infty}{\infty}\right|=\lim_{n\to\infty}\frac{n^3}{50n^3}=\frac{1}{50}.$$

It is obvious that $l \neq 0$, $l \neq \infty$, then according to the comparison test both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge simultaneously. The given series converges, because the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.

e) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$.

Consider the function that specifies the general term of the series:

$$f(n) = \frac{1}{(n+1)\ln(n+1)}.$$

It satisfies all the conditions of the corresponding theorem: it is positive and decreases monotonically. Therefore, the Cauchy integral test can be applied. To do this, we consider a definite improper integral of the first kind and test it for convergence:

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{dx}{(x+1)\ln(x+1)} = \int_1^{\infty} \frac{d(\ln(x+1))}{\ln(x+1)} =$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{d(\ln(x+1))}{\ln(x+1)} = \lim_{b\to\infty} \left(\ln[\ln(x+1)]\Big|_1^b \right) =$$

$$= \lim_{b\to\infty} \ln[\ln(b+1)] - \ln\ln 2 = \infty.$$

Since the limit is infinite, the considered improper integral of the first kind diverges, and therefore, according to the Cauchy integral, the original series also diverges.

Answer: a) $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{n^2+3}{n+2} = \infty$; b) $\sum_{n=1}^{\infty} \frac{n^2+1}{(2n-1)!} < \infty$;

c) $\sum_{n=1}^{\infty} \left(\frac{n+3}{n+1} \right)^{n^2} = \infty$; d) $\sum_{n=1}^{\infty} \frac{n\sqrt{n}}{5n^3 - 2n + 1} < \infty$;

e) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)} = \infty$.

Task 2. Test the following series $\sum_{n=1}^{\infty}(-1)^n a_n$ for absolute and conditional convergence, if a_n has the form:

$$a_n = \frac{1}{\sqrt[3]{n^2+7n-1}}\ .$$

Solution. So, we must test the alternating series:

$$\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n^2+7n-1}}.$$

First, consider the series of absolute values:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt[3]{n^2+7n-1}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2+7n-1}}.$$

It is clear that

$$a_n = \frac{1}{\sqrt[3]{n^2+7n-1}} \sim \frac{1}{\sqrt[3]{n^2}} = \frac{1}{n^{2/3}} = b_n.$$

So, in order to test the convergence of the series of absolute values we use the comparison test with the Dirichlet series:

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^p} = \sum_{n=1}^{\infty}\frac{1}{n^{2/3}}.$$

Here $p = \frac{2}{3} < 1$ that is, according to (11) we conclude that the series diverges.

Let us verify the conditions of Leibniz's theorem. Indeed,

1) $\lim\limits_{n\to\infty}\frac{1}{\sqrt[3]{n^2+7n-1}} = 0$;

2) $\frac{1}{\sqrt[3]{7}} > \frac{1}{\sqrt[3]{17}} > \frac{1}{\sqrt[3]{29}} > \ldots$,

Therefore, the series $\sum_{n=1}^{\infty} a_n$ converges according to Leibniz's theorem.

But since the terms of the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+7n-1}} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

are nonnegative and there exists

$$l = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2+7n-1}} \cdot \frac{n^{2/3}}{1} = 1,\ l \neq 0,\ l \neq \infty,$$

both series either converge or diverge simultaneously.

The series of absolute values $\sum_{n=1}^{\infty} |a_n|$ diverges on the basis of the limit comparison test. Thus, there is no absolute convergence. The given alternating series is conditionally converging.

Answer: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+7n-1}}$ is conditionally converging.

Individual tasks

Task 1. Test the numerical series $\sum_{n=1}^{\infty} a_n$ for convergence, if the general term a_n of this series has the form:

Variant 1.

a) $a_n = arcctg\,\frac{n-3}{n+2}$; b) $a_n = \frac{n+\sqrt{n}}{7^n}$;

c) $a_n = \left(\frac{2\pi n^2}{6n^2+n}\right)^{-n}$; d) $a_n = \frac{3n+1}{5n^2-2n}$;

e) $a_n = \frac{1}{n(4+\ln n)^3}$.

Variant 2.

a) $a_n = \sin\left(\frac{\pi}{2} + \pi n\right)$; b) $a_n = \frac{(1{,}1)^n}{n^2}$;

c) $a_n = \left(\ln\left(\pi + \frac{2}{n}\right)\right)^{-n}$; d) $a_n = \frac{2n\sqrt{n}+1}{3n^2+n}$;

e) $a_n = \frac{1}{(n^2+1)arctg^3 n}$.

Variant 3.

a) $a_n = \frac{n^2+1}{3n^2+n+2}$; b) $a_n = \frac{n!}{100^n}$;

c) $a_n = \left(arctg\frac{n}{2n^2+3}\right)^{\frac{n}{2}}$; d) $a_n = \frac{1}{\sqrt{n+1}(\sqrt{n}+5)}$;

e) $a_n = \frac{1}{n(\ln n + 9)}$.

Variant 4.

a) $a_n = \sqrt[3]{\frac{8n^2+1}{n^2+5n-1}}$; b) $a_n = \frac{e^n}{n3^n}$;

c) $a_n = \left(\cos\frac{\pi n}{3n+2}\right)^{-n}$; d) $a_n = \frac{2n^3+n}{n^5-n^2+6}$;

e) $a_n = \frac{1}{n\sqrt{1+\ln n}}$.

Variant 5.

a) $a_n = \ln\left(\frac{2n^2+3}{n^2+n}\right)$; b) $a_n = \frac{n^6+4n+7}{2^n}$;

c) $a_n = \left(\sin\frac{\pi n}{4n+1}\right)^{-2n}$; d) $a_n = \frac{\sqrt[3]{n+1}}{\sqrt{n^3+2n^3+5}}$;

e) $a_n = \frac{1}{n(\ln^2 n + 5)}$.

Variant 6.

a) $a_n = \sqrt[n]{e}$; b) $a_n = \frac{(n+1)!}{n^{100}}$;

c) $a_n = \left(\arccos\left(1+\frac{1}{n}\right)\right)^n$;   d) $a_n = \frac{n^2+2n}{3n^4-1}$;

e) $a_n = \frac{1}{n\sqrt{\ln n+1}}$.

Variant 7.

a) $a_n = \arccos\frac{1}{n}$;   b) $a_n = \frac{3^n}{n^2 2^n}$;

c) $a_n = \left(\frac{3n-1}{3n+5}\right)^{n^2}$;   d) $a_n = \frac{4\sqrt{n}+n}{3n^3+2}$;

e) $a_n = \frac{1}{n\ln(4n)}$.

Variant 8.

a) $a_n = \left(1+\frac{1}{n}\right)^{2n}$;   b) $a_n = \mathrm{tg}\,\frac{2}{3^n}$;

c) $a_n = (\ln\sqrt{n+6})^{-n}$;   d) $a_n = \frac{n^2\,\mathrm{arctg}(n+1)}{n^3+5}$;

e) $a_n = \frac{1}{(n^2+1)arctg^4 n}$.

Variant 9.

a) $a_n = \cos\frac{\pi}{n+1}$;   b) $a_n = \frac{n^2+3n}{(1{,}1)^{2n}}$;

c) $a_n = \left(\frac{5n+2}{4n+1}\right)^{\frac{n}{2}}$;   d) $a_n = \frac{1}{\sqrt{2n}(\sqrt{n}+3n)}$;

e) $a_n = \frac{1}{n(\ln n+1)^3}$.

Variant 10.

a) $a_n = \frac{n^2+5n-1}{100n^2+1}$;   b) $a_n = \sin\frac{1}{2^n}$;

c) $a_n = \left(\frac{n+1}{n+2}\right)^{n^2}$;   d) $a_n = \frac{\sqrt[3]{n+1}}{3n^2-1}$;

e) $a_n = \dfrac{1}{n\ln(3n)}$.

Variant 11.

a) $a_n = \dfrac{\sqrt[4]{n^2+1}}{\sqrt{4n+3}}$; b) $a_n = n^4\left(\dfrac{\pi}{4}\right)^n$;

c) $a_n = \left(\arccos\left(\dfrac{1}{n}\right)\right)^{-n}$; d) $a_n = \dfrac{n^2+n+7}{3n^3+2}$;

e) $a_n = \dfrac{1}{n(\ln n+3)^4}$.

Variant 12.

a) $a_n = \operatorname{arctg}\dfrac{n^2+2n+3}{n+1}$; b) $a_n = (n^2+2)e^{-n}$;

c) $a_n = \sqrt{\left(\dfrac{n^2+3}{4n^2-1}\right)^n}$; d) $a_n = (n+1)\sin\dfrac{1}{n^2+5}$;

e) $a_n = \dfrac{1}{(\ln^2 n+2)n}$.

Variant 13.

a) $a_n = n\sin\dfrac{1}{n}$; b) $a_n = \dfrac{10^n}{n!}$;

c) $a_n = \left(\dfrac{n+1}{2n-1}\right)^{-3n}$; d) $a_n = \dfrac{2n^3+1}{3n^4-4n}$;

e) $a_n = \dfrac{1}{n\sqrt[3]{2+\ln n}}$.

Variant 14.

a) $a_n = \left(\dfrac{6}{5}\right)^{\sqrt{n}}$; b) $a_n = \dfrac{n^3+3}{(2n+1)!}$;

c) $a_n = \left(\operatorname{tg}\left(\dfrac{\pi n}{3n+1}\right)\right)^{2n}$; d) $a_n = \dfrac{\sqrt{n+1}}{\sqrt{n^4+3n^2-1}}$;

e) $a_n = \dfrac{1}{n(\ln n+3)^2}$.

Variant 15. a) $a_n = \operatorname{tg}\frac{\pi n}{4n+9}$; b) $a_n = \frac{n^7+1}{n!}$;

c) $a_n = \left(\frac{2n+1}{2n+3}\right)^{-n^2}$; d) $a_n = \frac{\sqrt{2n^5+3}}{(n^2+1)^2}$;

e) $a_n = \frac{1}{(7\ln n+1)^2 n}$.

Variant 16. a) $a_n = (1{,}01)^{n+1}$; b) $a_n = \frac{n+3}{2^n(n-10)}$;

c) $a_n = \left(\frac{2n^2+1}{3n^2+5}\right)^{n}$; d) $a_n = \frac{1}{\ln n}$;

e) $a_n = \frac{1}{n\ln(5n)}$.

Variant 17. a) $a_n = \frac{n^n}{2n!}$; b) $a_n = \frac{n+4}{2^{n+1}n}$;

c) $a_n = \left(\frac{n+1}{2n+5}\right)^{n^2}$; d) $a_n = \frac{1}{(n+3)(n+5)}$;

e) $a_n = \frac{1}{n(3\ln n+7)}$.

Variant 18. a) $a_n = \frac{4^n}{(2^n+1)^2}$; b) $a_n = \frac{n^n}{n!}$;

c) $a_n = \left(\frac{n+1}{2n-4}\right)^{n^2}$; d) $a_n = \frac{n+2}{3(n^4+1)}$;

e) $a_n = \frac{1}{(n^2+1)arctgn}$.

Variant 19. a) $a_n = (2{,}5)^{n+1}$; b) $a_n = \frac{3^n+2^n}{n!}$;

c) $a_n = \left(\dfrac{2n+5}{2+n}\right)^{n^2}$; d) $a_n = \dfrac{arctg\sqrt{n^3-1}}{\sqrt{n^2+n}}$;

e) $a_n = \dfrac{1}{(\ln^2 n+1)n}$.

Variant 20.

a) $a_n = \dfrac{2n^2+1}{5n^2-3}$; b) $a_n = \dfrac{n}{3^n}$;

c) $a_n = \left(\dfrac{2n+2}{3n-1}\right)^{n^2}$; d) $a_n = \dfrac{\ln n}{\sqrt{n}}$;

e) $a_n = \dfrac{1}{(3\ln n+2)^4 n}$.

Variant 21.

a) $a_n = \left(1+\dfrac{1}{n}\right)^{n}$; b) $a_n = \dfrac{n^2+1}{2^n}$;

c) $a_n = \left(\dfrac{n+2}{10n+5}\right)^{n^2}$; d) $a_n = \dfrac{7n^2+n}{n^4+5n+16}$;

e) $a_n = \dfrac{1}{n(2\ln n+3)}$.

Variant 22.

a) $a_n = \dfrac{n\sqrt{4n^2+1}}{5n^2-3}$; b) $a_n = \dfrac{n!}{5^n}$;

c) $a_n = \left(\dfrac{n+1}{2n+5}\right)^{n^2}$; d) $a_n = \dfrac{4}{\sqrt{7n}(3n+\sqrt{n})}$;

e) $a_n = \dfrac{1}{n\sqrt{1+\ln^2 n}}$.

Variant 23

a) $a_n = \ln\left(\dfrac{5n^2-7}{n^2+n+1}\right)$; b) $a_n = \dfrac{n!}{n^{80}}$;

c) $a_n = \left(\dfrac{n+1}{4n-1}\right)^{n^3}$;   d) $a_n = \dfrac{n+8}{9n^2-3n}$;

e) $a_n = \dfrac{1}{n\sqrt[4]{3+\ln n}}$.

Variant 24.   a) $a_n = \sqrt[3]{\dfrac{27n^2+6}{n^2-5n+4}}$;   b) $a_n = \dfrac{n^2}{3^n}$;

c) $a_n = \left(\dfrac{4n+7}{5n+9}\right)^{n^3}$;   d) $a_n = \dfrac{\sqrt[3]{n-8}}{\sqrt{n^3+3n^2+2}}$;

e) $a_n = \dfrac{1}{(\ln n+3)^3 n}$.

Variant 25.   a) $a_n = \left(\dfrac{9}{8}\right)^{\sqrt{n}}$;   b) $a_n = \dfrac{n^n}{2^n n!}$;

c) $a_n = \left(\dfrac{2n+3}{5n-1}\right)^{n}$;   d) $a_n = \dfrac{1}{\sqrt{n}}\ln\left(\dfrac{n+1}{n-1}\right)$;

e) $a_n = \dfrac{1}{n(\ln n+2)}$.

Variant 26.   a) $a_n = \operatorname{arcctg}\dfrac{n+2}{n-3}$;   b) $a_n = \dfrac{\sqrt{n}+n}{6^n}$;

c) $a_n = \left(\ln\left(\dfrac{3}{n}+\dfrac{\pi}{2}\right)\right)^{-n}$;   d) $a_n = \dfrac{5n+4}{7n^2-3n+1}$;

e) $a_n = \dfrac{1}{n\sqrt[3]{\ln n+1}}$.

Variant 27.   a) $a_n = \dfrac{n^2+2n+3}{n+1}$;   b) $a_n = \dfrac{4^n}{3^n n^2}$;

c) $a_n = \left(\dfrac{7n+1}{7n-5}\right)^{n^2}$;   d) $a_n = \dfrac{n+5\sqrt{n}}{3n^3+2}$;

e) $a_n = \dfrac{1}{n\ln(2n)}$.

Variant 28.   a) $a_n = \left(\dfrac{1}{n}+1\right)^{5n}$;   b) $a_n = \operatorname{tg}\dfrac{3}{4^n}$;

c) $a_n = (\ln\sqrt{n+5})^{-n}$;   d) $a_n = \dfrac{n^3 \operatorname{arctg}(n+5)}{n^4+1}$;

e) $a_n = \dfrac{1}{(n^2+1)arctg^2 n}$.

Variant 29.   a) $a_n = \cos\dfrac{\pi}{n+2}$;   b) $a_n = \dfrac{(n^2+4)}{e^n}$;

c) $a_n = \left(\dfrac{2n+5}{n+4}\right)^{\frac{n}{2}}$;   d) $a_n = \dfrac{1}{\sqrt{n}(\sqrt{3n}+2n)}$;

e) $a_n = \dfrac{1}{(\ln n+3)^2 n}$.

Variant 30.   a) $a_n = \dfrac{n^2+n-5}{20n^2+7}$;   b) $a_n = n^5\left(\dfrac{\pi}{8}\right)^n$;

c) $a_n = \left(\dfrac{n+4}{n-3}\right)^{n^2}$;   d) $a_n = \dfrac{\sqrt[3]{n+1}}{4n^2+2}$;

e) $a_n = \dfrac{1}{n\sqrt{4+\ln n}}$.

Task 2. Test the following series $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ for absolute and conditional convergence, if a_n has the form:

Variant 1. $$a_n = \frac{2n+5}{(3n-1)(n+1)}.$$

Variant 2. $$a_n = \frac{2n^2-1}{n!}.$$

Variant 3. $$a_n = \frac{1}{\ln(n+7)}.$$

Variant 4. $$a_n = \frac{n^3+4}{(n+1)!}.$$

Variant 5. $$a_n = \frac{1}{\sqrt[3]{n^2+n+9}}.$$

Variant 6. $$a_n = \frac{n^2+3}{5^{n+1}}.$$

Variant 7. $$a_n = \frac{1}{(2n+1)\sqrt{n+3}}.$$

Variant 8. $$a_n = \frac{n+8}{(2n-1)!}.$$

Variant 9. $$a_n = \frac{1}{7+\sqrt[4]{2n+1}+\sqrt{n}}.$$

Variant 10. $$a_n = \frac{6n-1}{n^2+3n+2}.$$

Variant 11. $$a_n = \frac{\sqrt[3]{n^2}}{4+3n+n\sqrt{n}}.$$

Variant 12. $$a_n = \frac{n^2+3}{(2n+1)!}.$$

Variant 13. $$a_n = \frac{n}{\sqrt{2n^3 + 3n^2 + 1}}.$$

Variant 14. $$a_n = \frac{n^2 + 5}{5^{n-1}}.$$

Variant 15. $$a_n = \frac{3^{n+1}}{n^2 + 3}.$$

Variant 16. $$a_n = \frac{n+2}{n^2 + 4}.$$

Variant 17. $$a_n = \frac{n}{5^n}.$$

Variant 18. $$a_n = 1 - \cos\frac{1}{\sqrt{n}}.$$

Variant 19. $$a_n = arctg\frac{1}{n}\sin\frac{1}{n}.$$

Variant 20. $$a_n = \frac{2n+1}{7n+3}.$$

Variant 21. $$a_n = \frac{1}{n\ln(3n)}.$$

Variant 22. $$a_n = \frac{n^4 + 5}{(n+2)!}.$$

Variant 23. $$a_n = \frac{1}{\sqrt[4]{n+1} + 2\sqrt{n}}.$$

Variant 24. $$a_n = \frac{1}{\ln(5+n)}.$$

Variant 25. $a_n = \dfrac{1}{\sqrt[5]{2n+1}-3}$.

Variant 26. $a_n = \dfrac{3n^2+2}{15^{n+1}}$.

Variant 27. $a_n = \dfrac{1}{\sqrt{n+3}(7n+1)}$.

Variant 28. $a_n = \dfrac{n-6}{(3n+1)!}$.

Variant 29. $a_n = \dfrac{1}{\sqrt[3]{2n+1}+\sqrt{n}+4}$.

Variant 30. $a_n = \dfrac{8n+1}{n^2-4n-2}$.

Bibliography

1. Англо-русский словарь математических терминов / под ред. П. С. Александрова. – Москва : Мир, 1994. – 416 с.

2. Берман Г. Н. Сборник задач по курсу математического анализа / Г. Н. Берман. – Москва : Наука, 2002. – 384 с.

3. Васильченко Г. П. Вища математика для економістів : підручник / Г. П. Васильченко. – Київ : Знання – Прес, 2002. – 454 с.

4. Вища математика : базовий підручник для вузів / під ред. В. С. Пономаренка. – Харків : Фоліо, 2014. – 669 с.

5. Данко П. Е. Высшая математика в упражнениях и задачах. В 2-х ч. / П. Е. Данко, А. Г. Попов, Т. Я. Кожевникова. – Москва : Высшая школа, 2003. – 304 с. и 416 с.

6. Контрольні завдання з курсу "Математика для економістів" для студентів усіх форм навчання. У 2-х ч. Ч. 1 / уклад. Е. Ю. Железнякова, А. В. Ігначкова, Л. Д. Широкорад. – Харків : Вид-во ХНЕУ, 2005. – 52 с.

7. Красс М. С. Математика для экономических специальностей : учебник / М. С. Красс. – Москва : ИНФРА-М, 1999. – 464 с.

8. Малярець Л. М. Математика для економістів : практич. посіб. У 2-х ч. Ч. 2 / Л. М. Малярець, Л. Д. Широкорад. – Харків : Вид-во ХНЕУ, 2008. – 476 с.

9. Borakovskiy A. Handbook for problem solving in higher mathematics / A. Borakovskiy, A. Ropavka. – Kharkiv : KNMA, 2008. – 195 p.

10. Handbook of mathematics / I. Bronshtein, K. Semendyaev, G. Musiol et al. – Berlin : Springer, 2007. – 1097 p.

11. Higher mathematics : handbook. In 4 vol. Vol. 4 / ed. by L. Kurpa. – Kharkiv : NTU "KhPI", 2006. – 540 p.

Content

НАВЧАЛЬНЕ ВИДАННЯ

ВИЩА МАТЕМАТИКА

Методичні рекомендації до самостійної роботи за темою "Ряди" для студентів галузі 12 "Інформаційні технології" першого (бакалаврського) рівня (англ. мовою)

Самостійне електронне текстове мережеве видання

Укладачі: **Рибалко** Антоніна Павлівна
Стєпанова Катерина Вадимівна

Відповідальний за видання *Л. М. Малярець*

Редактор *З. В. Зобова*

Коректор *З. В. Зобова*

Викладено необхідний теоретичний матеріал з навчальної дисципліни та наведено типові приклади, які сприяють найбільш повному засвоєнню матеріалу з теми "Ряди" та застосуванню отриманих знань на практиці. Наведено детальний опис та методичні рекомендації до виконання завдань для самостійної роботи, перелік літературних джерел, запитання та тест для самодіагностики з метою вдосконалення знань студентів за даною темою. Визначено професійні компетентності, яких набувають студенти внаслідок вивчення теоретичного матеріалу й виконання практичних завдань за цією темою.

Рекомендовано для студентів галузі 12 "Інформаційні технології" першого (бакалаврського) рівня.

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