Jornal homepage: https://devma.com.ua/en

DEVELOPMENT MANAGEMENT

UDC 330.3:330.4
DOI: 10.57111/devt.20(2).2022.8-15

Vol. 20, No. 2. 2022

A Nonlinear Mathematical Model of Dynamics of Production and Economic Objects

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Abstract. A person who makes decisions regarding the management of an industrial and economic object feels the need for tools to solve numerous problems that arise in the process of functioning of this object as an economic system in the conditions of interaction with the environment. The purpose of this work is to build an analytical model of the production and economic system, which would allow studying the structural changes that may occur in the process of functioning of economic objects with a closed cycle of production activity and which determine the possible ways of evolution of an open economic system over time (phase trajectories of evolution). The methodology of nonlinear dynamics and economic synergy was used to create the model. The work proposes a mathematical model of the production and economic system with a small number of phase variables that have a market interpretation, and determines endogenous and exogenous parameters that characterize the state of the system and the direction of its development. The model contains a system of two ordinary differential equations with quadratic nonlinearity. This formalization made it possible to obtain general information about the development trajectories of this system and its stationary states with the identification of the most significant critical modes of functioning. Qualitative analysis based on this model showed that non-linearity leads to non-unity of equilibrium states and the existence of both stable and unstable development trajectories of the economic system under study. This model can be used to manage any complete economic unit in which an independent closed cycle of reproduction is ensured

Keywords: economic dynamics, mathematical model in continuous time, nonlinear dynamics, synergism, phase trajectories of evolution, stability of equilibrium points, bifurcation

Article's History: Received: 28.02.2022; Revised: 11.05.2022; Accepted: 07.06.2022

INTRODUCTION

The paradigm shift that took place in economic theory during the last decades led to the formation of such a scientific methodology as economic synergy. To replace the concept of linearity of dynamic processes, there is an awareness of the need to take into account nonlinear effects associated with the existence of real large feedback systems, as well as the presence of critical levels of parameter values characterizing the state of the system. The effect of nonlinearity is manifested in the fact that instead of a single solution that would uniquely determine the development of the system over time in accordance with the initial conditions, a whole spectrum of solutions appears. It is expedient to consider a modern enterprise as a complex production and economic system, which is characterized by non-linear interaction of processes that determine the state of this system. When developing a mathematical model of a production and economic system, attention should be paid to such system properties. First of all, it is manifested in a large number of various structural elements that are interconnected and constantly interact with each other. In turn, each of these structural elements is also a complex system. These structural elements have a different nature and can be considered as separate open systems. That is, a complex system (industry, holding, enterprise) can be provided as a set of subsystems. At the same time, the purpose of functioning of each of the subsystems is subordinated to the purpose of functioning of the system as a whole. It is this set of structural components in their interaction that determines the behavior of the system, which is manifested in a series of changes in its states (development trajectories) over time.

Suggested Citation:
Voronin, A., Lebedeva, I., & Lebedev, S. (2022). A nonlinear mathematical model of dynamics of production and economic objects. *Development Management,* 20(2), 8-15.

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The interaction of some structural elements of the system can be additive (in this case, the system is linear), but for the system as a whole, a synergistic effect is observed (this is a manifestation of nonlinearity). In addition, complex dynamic systems are open, that is, the functioning of the system occurs in the interaction of its structural elements not only among themselves, but also with the external environment. It is under the influence of the external environment and internal processes caused by this influence that the system transitions from one state to another, that is, quantitative changes in the parameters characterizing the state of the system are observed. Another complication is that, in general, the production and economic system is under the influence of random factors, so the functioning of any production has a probabilistic nature. It should also be noted that in such complex systems there may be feedback loops, that is, there may be chains of cause-and-effect relationships, according to which the output data partially returns back as input data. It follows from this that the person who makes a decision regarding the management of the development of the production and economic system should do so on the basis of a thorough analysis of the possible trajectories of the system's development and the determination of the factors influencing the direction of this development. Such opportunities are provided by mathematical modeling, which is one of the most effective research methods. In this sense, mathematical modeling is a universal tool for carrying out qualitative analysis (analysis of possible trajectories of system development), which allows predicting the system's behavior over time. The need to take into account a large number of variable factors and parameters when building a mathematical model of the production and economic system leads to an increase in the dimensionality of the model. This significantly complicates the quantitative verification of possible scenarios of the development of production and economic systems and its possible equilibrium states, even despite the high development of computer technology.

Therefore, when creating a mathematical model of the production and economic system, it is necessary to limit the number of endogenous parameters, leaving only those of them that most significantly affect the development of the system, as well as to highlight the most general so-called order parameters that reflect the interaction of the system with the external environment. This approach makes it possible to analyze the potential trajectories of system development and to determine the leading factors of influence, the management of which makes it possible to realize the equilibrium states of the system or to direct the development of the system along the selected trajectory.

The purpose of this work is to create an analytical model that would allow research into possible trajectories of the evolution of the production and economic system based on the methodology of nonlinear dynamics and economic synergy. The approach to creating a mathematical model of the dynamics of the production and economic system, proposed by the authors, combines the solution of this problem with the help of technical and economic analysis and qualitative forecasting (according to phase trajectories).

LITERATURE REVIEW

Within the paradigm of linear economy, production systems are considered as systems whose purpose is to meet the needs of consumers, and their structure is determined by the dependence on available resources and demand for products. Accordingly, the operation of such a system is determined by the principle of "take – do – throw". In contrast to this systems thinking, which is inherent in the economics of complex dynamical systems, allows us to study the structure of such systems, asking "what if" about the potential behavior of the system, thereby improving understanding of causal behavior and its transition to a state that previously seemed paradoxical [1].

If system thinking is a way of describing and understanding cause-and-effect relationships within the system, then system dynamics as a research method is the construction of a mathematical model, which in its final implementation gives a qualitative idea of the result of the interaction of these relationships. A problem-oriented approach to modeling (creating models of system dynamics) was proposed by Jay Forrester in the late 1950s. His research was aimed at helping corporate managers to better understand production problems, so the principles of nonlinear system dynamics became an approach to the development of corporate policy [2]. If traditional research methods suggest solving a complex problem by breaking it down into separate components and examining each of them separately, then system dynamics is based on the idea that problems exist precisely because of the interaction of the structural components of the system. The peculiarity of this interaction is that it gives a synergistic effect. This can lead to the formation of feedback loops, information lag within the system, etc. At the current stages of the development of economic science, it is the system dynamics that is the basis of the choice of methods for finding solutions to such problems. This approach has found its application in various scientific fields. Examples of its use include modeling processes responsible for improving the health care system [3], when solving sociological problems [4; 5], studying the processes that determine the development of micro- and macroeconomics [6-8], solving complex problems of regional development [9-11], for building models of performant development [12], etc. This approach is the basis of a new direction in analytical economics, which was named synergistic economics [13]. Within the framework of this direction, special aspects of the nonlinearity of the development of complex systems are considered. These include bifurcations, hysteresis, as well as chaos (catastrophe), when small changes in system parameters can lead to different results, or cases when the development of the system leads to the formation of an increasing number of structural patterns [14; 15]. During economic analysis, sudden structural changes, the presence of regular and irregular fluctuations, the role of random factors in economic evolution, as well as the influence of time scales and rates of adjustment of economic variables on the processes of system development are of particular interest.

The concept of "synergy", as well as the mathematical apparatus used to model such systems, came to economics from physics back in the 60s and 70s of the

20th century. This is due to the fact that studies of open economic systems revealed a violation of additivity when moving from the analysis of the activity of individual elements of the system to the determination of the characteristics of the entire system as a whole. That is, the total effect of the interaction of the structural components of a complex system (its subsystems) significantly exceeds the effect of simply adding the corresponding characteristics of each of the components, if they develop autonomously, without interacting with each other. Such complex systems are defined as nonlinear. For non-linear time series, the use of traditional econometric analysis is impossible. In this case, the theoretical foundations of economic synergy are used as a scientific methodology, which studies the self-organization of developing systems, when this development is based on multiplicative processes. Therefore, the most important thing in the study of such systems is the so-called qualitative forecasting, that is, the analysis of the hierarchy of stable and unstable trajectories with the classification of special points (equilibrium positions). At the same time, the exact value of the parameters of the economic system is not mandatories; it is enough that they are comparable.

In mathematical modeling, system dynamics is considered as an approach that allows describing the structure of complex systems, taking into account material and informational feedback loops. This makes it possible to investigate the causes and consequences that cause turbulence in development trajectories, and to develop highly effective management solutions aimed at increasing the productivity of the entire system as a whole [16]. Schematically, the modeling process can be represented as a sequence of such stages (Fig. 1).

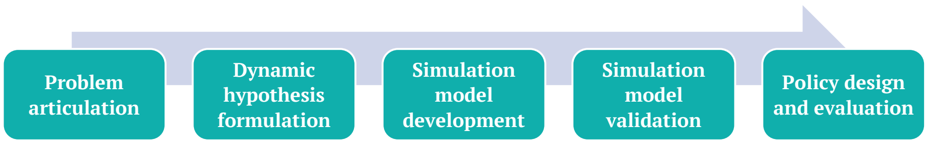

Figure 1. Stages of creating a system dynamics model

Source: developed by the authors based on research [16]

Creating models of economic dynamics consists of two generalized stages: qualitative and quantitative [17]. Initially, the problem must be formulated in terms of the choice of research topic, key variables, time horizon, and the history of behavior of key concepts and variables. Then a working hypothesis is formulated to reflect an understanding of how the researcher perceives the key causal mechanisms that give rise to the problem. This is a qualitative stage of building a dynamic system model. It is now possible to proceed to the quantitative stage, in which a simulation model is developed to test the ability of the dynamic hypothesis to determine answers to key questions about the functioning of the system. Finally, the verified dynamic model is used to develop a control influence that will determine the development of the system along the selected trajectory.

Forrester's key idea, which is the basis of the modeling of economic systems of dynamics, was that a mathematical model of any complex system can be created using only two types of variables: stocks and flows [2]. Stocks can only be changed through their associated inputs and outputs. Flow regulation, in turn, is carried out using functions that act similarly to valves located at the inlet and outlet. Due to the presence of closed loops and time lags in such systems, elements of nonlinearity behavior arise. As a result, such development trajectories appear that are absent in linear systems. Examples of trajectories that are inherent only to nonlinear systems can be sub- and superharmonic phenomena, synchronization, bifurcations (as a result of which significant jumps in development occur), hysteresis (ambiguous dependence of changes in the state of the system on changes in external factors), and chaos (when small changes in the initial conditions can lead to completely different results). The most interesting in this regard is the self-organization of the system, that is, the formation of a new ordered structure as a result of the evolutionary and adaptation processes taking place in the system.

The description of the evolution of the states of a complex system within the framework of nonlinear dynamics requires the use of a certain formal apparatus. As a rule, system of differential equations is used for this, if the studied process is continuous in time. Or, if time is considered discrete, system of equations in finite differences is used. The main complication that arises in the study of nonlinear systems is that the principle of superposition (which assumes the additivity of the effects of various factors) does not apply to such systems. It should be noted that this principle allows for any linear problem to separate the linear equations describing the system evolution, and this cannot be done for a nonlinear system. To overcome this shortcoming, several methods have been proposed in recent decades for finding the solution of a system of differential equations describing the state of nonlinear systems. The toolkit for development of mathematical models of nonlinear dynamic systems and means of their analysis are becoming more and more important for solving real problems not only in technical sciences, but also in sociology, economics [14; 15], biology etc.

MATERIALS AND METHODS

In this study, when creating a mathematical model of a complex production and economic system, time is considered as a system-forming factor. The analysis of the evolution of such a system consists in the fact that its development is considered as a movement in phase space along a certain phase trajectory, that is, the analysis carried out is qualitative. The task of such an analysis is to study how the qualitative restructuring of topological structure of the system takes place when parameters of system are changed. When developing a mathematical model of the production and economic system, it is necessary to limit the number of endogenous parameters by determining the most common of them (the so-called order parameters), which reflect the interaction of the system with its external environment under the influence of various exogenous factors.

To determine the state of the dynamic system, it is necessary to take into account the following elements [18]. First of all, it is the metric space *U* which is defined as a phase space. Secondly, it is a variable *t* that characterizes time. In this paper, time is considered continuous, i.e. $t \in R^1$. It should be emphasized that such a mathematical model determines the evolution of the dynamic system in real time, that is, it describes a dynamic system which is continuous in time. Thirdly, it is the law of evolution, that is, a function that for any point of the phase space *U* and any value of *t* uniquely reflects the state of the system $\varphi(t, X) \in U$. This function has the following properties:

$$\varphi(0,\ X) = X;$$
$$\varphi\big(t_1,\ \varphi(t_2,\ X)\big) = \varphi(t_1 + t_2,\ X),$$

where the function $\varphi(t, X)$ is continuous in its arguments (X, t).

To determine the phase trajectories, along which the evolution of the system is carried out, the tools of differential calculus were used. The evolving system can be mathematically described by a system of differential equations, which in standard form look like this:

$$\frac{dX}{dt} = \dot{X} = \varphi(t,\ X). \qquad (1)$$

In the problem considered in this study, the vector X has two components (unit price and demand volume), i.e. $X = X(t) \in R^2$, and the function $\varphi(X, t)$ is a sufficiently smooth function defined by to some subset $U \subseteq R^2 \times R$.

Since the production and economic system is dissipative, integration was not used, but a qualitative analysis of the system of differential equations (analysis by phase trajectories) was carried out. Their stability was considered one of the main issues that were paid attention to during the study of the properties of possible development trajectories.

RESULTS

The production and economic system can use its profit for the needs of production development and for non-production accumulations:

$$E = I + S, \qquad (2)$$

where *E* is the amount of profit; *I* is volume of investment in production; *S* is savings.

Savings are made for the purpose of further use of financial resources for the formation of a reserve fund, conducting scientific research, research and development works, etc. Thus, savings related to the factors contributing to changes in the structure of production and, accordingly, the trajectory of the development of the production and economic system [19]. Therefore, since part of the profit from the activity of the production and economic system is invested in production, that is, the output data is partially transferred to the input, feedback is realized in the system.

Suppose that the growth of demand for production products is determined by a linear decreasing function:

$$D = d_0 - a \cdot p, \qquad (3)$$

where $D = D(p)$ is the market demand function; $p = p(t)$ is the price of a unit of production; d_0 and a are the corresponding parameters of the demand function.

Let's assume that the volume of products produced during the studied period is sold on the markets without the formation of stocks. In this case, the price regulation mechanism is determined by the difference between the demand and supply of products:

$$\frac{dp}{dt} = \beta\left(b - \frac{y}{a} - p\right), \qquad (4)$$

where the parameter b is determined by the correlation $b = \frac{d_0}{a}$; $y = y(t)$ is a function that describes the amount of supply at the relevant time; β is the coefficient of market adaptation.

Profit from sold products can be described by the formula:

$$E = (p - c)y - c_0, \qquad (5)$$

where c is conditional variable costs; c_0 is conditionally constant costs.

We will assume that the amount of savings grows in proportion to the amount of profit:

$$S = sE,\ 0 \le s \le 1, \qquad (6)$$

where *s* is the savings multiplier.

According to equations (2) and (6), we have:

$$I = (1 - s)E. \qquad (7)$$

The increase in the volume of production is carried out at the expense of the investment component of profit:

$$y = \gamma I,\ \gamma > 0,$$

or

$$\frac{dy}{dt} = \alpha \cdot ((p - c) \cdot y - c_0), \qquad (8)$$

where $\alpha = \gamma \cdot (1 - s)$ is the marginal cost of increasing output.

Differential equations (4) and (8) completely determine the dynamics of the production and economic system, where the volume of production and the price of a unit of production appear as order parameters:

$$\begin{cases} \frac{dy}{dt} = \alpha((p - c)y - c_0); \\ \frac{dp}{dt} = \beta\left(b - \frac{y}{a} - p\right). \end{cases} \qquad (9)$$

Such a system of two ordinary differential equations has special solutions that satisfy the algebraic equation:

$$\begin{cases} p^2 - (b - c)p + bc + \frac{c_0}{a} = 0; \\ y = a(b - p). \end{cases} \qquad (10)$$

It follows from system (10) that the condition for the existence of two special solutions is the fulfillment of the inequalities:

$$\begin{cases} b - c > 0; \\ a\left(\frac{b-c}{2}\right)^2 > c_0. \end{cases} \qquad (11)$$

And the coordinates of special points corresponding to the equilibrium state of system (9) are calculated using the following formulas:

$$p^*_{1,2} = \frac{b+c}{2} \pm \sqrt{\left(\frac{b-c}{2}\right)^2 - \frac{c_0}{a}}, \quad p^*_1 < p^*_2$$
$$y^*_{1,2} = a\left(\frac{b-c}{2} \mp \sqrt{\left(\frac{b-c}{2}\right)^2 - \frac{c_0}{a}}\right), \quad y^*_1 > y^*_2$$

It is convenient to study the properties of system (9) using deviations from the equilibrium values p^* and y^*, and also in the new time scale $t_0 = \alpha \cdot t$. Then this system takes the form:

$$\begin{cases} \frac{d\tilde{y}}{dt} = (p^* - c)\tilde{y} + a(b - p^*)\tilde{p} + \tilde{y} \cdot \tilde{p}; \\ \frac{d\tilde{p}}{dt} = -\frac{\xi}{a}\tilde{y} - \xi\tilde{p}, \end{cases} \tag{12}$$

where $\tilde{p} = p - p^*$, $\tilde{y} = y - y^*$ and $\xi = \frac{\beta}{\alpha}$.

The matrix of the linear part of the system (12) has the structure:

$$\mathbf{A} = \begin{pmatrix} p^* - c & a(b - p^*) \\ -\frac{\xi}{a} & -\xi \end{pmatrix}.$$

This matrix has a characteristic polynomial:

$$\lambda^2 - \mathrm{tr}\mathbf{A} \cdot \lambda + \det \mathbf{A} = 0, \tag{13}$$

where $\mathrm{tr}\mathbf{A} = p^* - c - \xi$ is the trace of the matrix $\mathbf{A}$; $\det \mathbf{A} = \xi(b + c - 2p^*)$ is the determinant of the matrix $\mathbf{A}$.

Thanks to the consideration of the characteristic equation (13), we obtain the stability conditions of the equilibrium position of the system (12), which can be given in the form:

$$\begin{cases} p^* - c - \xi < 0; \\ b + c - 2p^* > 0. \end{cases} \tag{14}$$

For the lowest equilibrium unit price p_1^*, which is the first of the inequalities of system (14), gives the relation:

$$\xi > \frac{b-c}{2} - \sqrt{\left(\frac{b-c}{2}\right)^2 - \frac{c_0}{a}},$$

and the second inequality of system (14) is fulfilled automatically.

For a special point p_2^*, which is greater than p_1^*, the inequality $b - c < 2p_2^*$ always holds. Therefore, this state of equilibrium is unstable for any ratio of system parameters (12). And the observed instability is saddle.

The analysis of the mathematical expressions for the trace and the determinant of the matrix of the system (12) showed that for each of them, independently of each other, a change of sign is possible. Therefore, let us assume that the trace tr**A** and the determinant det**A** of this matrix are small sign variables, i.e.

$$\begin{gathered} c + \xi - p^* = \mu_2, \\ \xi(b + c - 2p^*) = \mu_1, \end{gathered} \tag{15}$$

where μ_1 and μ_2 are small parameters.

With the help of relations (10), we exclude the coordinate of the equilibrium price p^* from the expressions (15) and obtain the connection equation for determining the bifurcation parameters of the system (12):

$$\begin{aligned} c_0 &= a(\xi - \mu_2)\left(\xi - \mu_2 + \frac{\mu_1}{\xi}\right), \\ b - c &= 2(\xi - \mu_2) + \frac{\mu_1}{\xi}. \end{aligned} \tag{16}$$

Taking into account the obtained ratios (16), system (12) takes the form:

$$\begin{cases} \frac{d\tilde{y}}{dt} = (\xi - \mu_2)\tilde{y} + a\left(\xi - \mu_2 + \frac{\mu_1}{\xi}\right)\tilde{p} + \tilde{y} \cdot \tilde{p}; \\ \frac{d\tilde{p}}{dt} = -\frac{\xi}{a}\tilde{y} - \xi\tilde{p}, \end{cases} \tag{17}$$

and, accordingly, we will have the characteristic equation (13) in the form:

$$\lambda^2 + \mu_2\lambda + \mu_1 = 0. \tag{18}$$

When $\mu_1 = \mu_2 = 0$, the solution of equation (18) is twice zero, so it can be assumed that in the nonlinear system of differential equations (17) the so-called Bogdanov-Takens bifurcation may arise, for which it is necessary to carry out a variation of two parameters [20; 21]. Such a bifurcation can occur if the linearization of the function around a stationary point has a double eigenvalue at zero. It should be noted that knowing the location of the bifurcation points and the type of bifurcation that is realized at this point is of great importance, as it marks the transition from one dynamic mode to another.

For a detailed study of the properties of the bifurcation of the "double zero", i.e. Bogdanov-Takens bifurcations, it is necessary to present the system (17) as the corresponding normal form. For this purpose, we will introduce new variables. Let $\tilde{y} = -\xi x_1 - x_2$ and $\tilde{p} = \frac{\xi}{a}x_1$. Then, after algebraic transformations, this system can be written in the form:

$$\begin{cases} \frac{dx_1}{dt} = x_2; \\ \frac{dx_2}{dt} = -\mu_1 x_1 - \mu_2 x_2 + \frac{\xi^2}{a}x_1^2 + \frac{\xi}{a}x_1 x_2. \end{cases} \tag{19}$$

Let's make another substitution of variables:

$$x_1 = a\left(y_1 + \frac{\mu_1}{2\xi^2}\right), \quad x_2 = a\xi y_2.$$

With the help of the new time scale , we obtain the desired normal form for the system of differential equations (17):

$$\begin{cases} \frac{dy_1}{dt} = y_2; \\ \frac{dy_2}{dt} = \beta_1 + \beta_2 y_2 + y_1^2 + y_1 y_2, \end{cases} \tag{20}$$

where $\beta_1 = -\frac{\mu_1^2}{4\xi^4}$; $\beta_2 = \frac{\mu_1}{2\xi^2} - \frac{\mu_2}{\xi}$.

So, we obtained a system of differential equations in the standard form (1). Hence, it is not difficult to find bifurcation curves on which system (20) has a "saddle-node" bifurcation and a Hopf bifurcation. The "saddle-node" bifurcation is characterized by the fact that only one bifurcation curve from singular points passes through the bifurcation point. A Hopf bifurcation is a local bifurcation when a stationary point of a dynamical system loses stability, and this loss of stability leads to the appearance of periodic solutions.

First, we note that stationary points (equilibrium states) are given by the relation:

$$(y_1^*; \ \ y_2^*) = \left(\pm\sqrt{-\beta_1}\ ; \ \ 0\right). \tag{21}$$

They have always existed since $\beta_1 < 0$. Linearization around these points leads to the expression:

$$\mathbf{F} = \begin{pmatrix} 0 & 1 \\ \pm\sqrt{-\beta_1} & \beta_2 \pm \sqrt{-\beta_1} \end{pmatrix}.$$

It follows that the point $\left(+\sqrt{-\beta_1};\ 0\right)$ is stable, and the point $\left(-\sqrt{-\beta_1};\ 0\right)$ is a source when $\beta_2 > \sqrt{-\beta_1}$ or a drain when $\beta_2 < \sqrt{-\beta_1}$. Thus, the Hopf bifurcation takes place on the curve $\beta_2 = \sqrt{-\beta_1}$, and the bifurcation "saddle-node" is realized on the plane $\beta_1 = 0$ if $\beta_2 \neq 0$.

To study the stability of the Hopf bifurcation, we will make two substitutions of variables, the first of which allows us to reduce the vector field to a standard form. We will assume that $\bar{y}_1 = y_1 + \sqrt{-\beta_1}$ and $\bar{y}_2 = y_2$. In this case, we get:

$$\begin{cases} \frac{d\bar{y}_1}{dt} = \bar{y}_2; \\ \frac{d\bar{y}_2}{dt} = -2\sqrt{-\beta_1}\cdot\bar{y}_1 + \bar{y}_1\cdot\bar{y}_2 + \bar{y}_1^2. \end{cases} \quad (22)$$

Now let's use the linear transformation $\bar{y}_1 = u_2$ and $\bar{y}_2 = \sqrt{2\sqrt{-\beta_1}}\cdot u_1$, the matrix of which consists of the real and imaginary parts of the eigenvectors corresponding to the eigenvalues of this matrix: $\lambda_{1,2} = \pm i\sqrt{2\sqrt{-\beta_1}}$. Thanks to this, we get a system of differential equations, the linear part of which is written in the standard form:

$$\begin{cases} \frac{du_1}{dt} = -\sqrt{2\sqrt{-\beta_1}}\cdot u_2 + u_1\cdot u_2 + \frac{1}{\sqrt{2\sqrt{-\beta_1}}}u_2^2; \\ \frac{du_2}{dt} = \sqrt{2\sqrt{-\beta_1}} + u_1. \end{cases} \quad (23)$$

For system (23), the first Lyapunov quantity [22], which characterizes the stability of the limit cycle, has the following form:

$$l_1 = \frac{1}{16\sqrt{-\beta_1}} > 0. \quad (24)$$

Note that the Lyapunov quantities characterize how well the system "remembers" the initial state, that is, they determine the local stability and instability of a weak focus. A positive Lyapunov indicator shows how quickly points located next to each other diverge. The negative Lyapunov indicator shows how quickly the system recovers after an external impact, that is, it determines the time required for the system to recover the limit cycle. Accordingly, the Hopf bifurcation is subcritical, and we have a family of unstable periodic orbits surrounding the flow (stable focus) when the value of the parameter β_2, is less than $\sqrt{-\beta_1}$, but close to this value.

Next, we determine whether a global bifurcation occurs. Perhaps this is a loop of a saddle-focus separatrix in which the limit cycle disappears and the stable and unstable manifolds of the saddle point "cross". To study such a bifurcation, we apply the scale transformation:

$$y_1 = \varepsilon^2 v_1; \quad y_2 = \varepsilon^2 v_2; \ \beta_1 = \varepsilon^4\alpha_1; \quad \beta_2 = \varepsilon^2\alpha_2; \quad \varepsilon \geq 0\,. \quad (25)$$

We will also introduce a new variable that characterizes the time $\tau \to \varepsilon\cdot\tau_0$. And system (19) takes the form:

$$\begin{cases} \frac{dv_1}{dt} = v_2; \\ \frac{dv_2}{dt} = \alpha_1 + \varepsilon\alpha_2 v_2 + \varepsilon v_1 v_2 + v_1^2. \end{cases} \quad (26)$$

Assume $\varepsilon = 0$. Then, with a fixed value of the parameter $\alpha_1 \neq 0$, system (26) can easily be transformed into a Hamiltonian system, which is a particular case of a dynamic system and is characterized by the fact that it does not have dissipation:

$$\begin{cases} \frac{dv_1}{dt} = v_2; \\ \frac{dv_2}{dt} = \alpha_1 + v_1^2. \end{cases} \quad (27)$$

And this system has a Hamiltonian, i.e. fixed income:

$$H(v_1,\ v_2) = \frac{v_2^2}{2} - \alpha_1 v_1 - \frac{v_1^3}{3}.$$

This transformation makes it possible to perform integration. Now it becomes clear the motivation for the scale changes, which were made according to relations (25). We can perform a perturbation of the global phase curves of the system of differential equations (27), and this will allow us to determine the behavior of the system (20) for the case when β_1 and β_2 are close to zero. The search for saddle loops is reduced to the search for values of α_2 and $\varepsilon\approx 0$, for which a saddle connection is realized. Such a problem can be solved using Melnikov's method [20]. The solution is given by the formulas:

$$v_1^0(\tau_0) = 1 - 3\,sech^2\left(\frac{\tau_0}{\sqrt{2}}\right),$$
$$v_2^0(\tau_0) = 3\sqrt{2}\,sech^2\left(\frac{\tau_0}{\sqrt{2}}\right) th\left(\frac{\tau_0}{\sqrt{2}}\right). \quad (28)$$

In this case, the Melnikov's function $M(\tau_0)$ is stationary, and it can be specified as follows:

$$M(\alpha_2) = \int_{-\infty}^{\infty} v_2^0(t)(\alpha_2 v_2^0(t) + v_1^0(t))\,dt.$$

After integration, we equate the Melnikov function to zero and obtain the corresponding value $\alpha_2 = \frac{5}{7}$. This gives an approximate formula for the bifurcation curve in terms of parameters β_1 and β_2:

$$\beta_1 = -\frac{49}{25}\beta_2^2; \quad \beta_2 \geq 0.$$

The real bifurcation line is tangent to the given semiparabola at the point $\beta_1 = \beta_2 = 0$. In addition, it is essential that the trace of the linearization matrix is positive:

$$\mathrm{tr}\mathbf{F} = \beta_2 + \sqrt{-\beta_1} = \frac{12}{5}\beta_2 > 0\cdot$$

So, it was founded out that the production and economic system (9) has two equilibrium positions. There are a compound focus and saddle. The above qualitative analysis (by trajectories) of the structural stability of this system describes to a situation where both equilibrium positions are very close to each other. If the parameter that characterizes the level of conditionally constant costs increases to its critical value $c_0^* = \frac{a(b-c)^2}{4}$, both equilibrium positions merge with each other and then disappear. Therefore, the system loses its stability, which is called a "fold" in the phase space. In the case when the dynamic parameter ξ is close to $\xi^* = \frac{b-c}{2}$, in system (9) a complex focus gives birth to an unstable limit cycle, which is characterized by a rigid regime of self-oscillations.

DISCUSSION

The mathematical model of production and economic object proposed in this paper makes it possible to analyze the qualitative behavior of such nonlinear dynamic systems when changing the parameters which characterize these systems, allows describing states that are far from equilibrium, and also makes it possible to predict a sharp change in the state of the system when a slight change in its parameters. It is appropriate to compare the obtained results with the data published in the paper of K. Sasakura [23], as well as in the paper of G. Feichtinger [24], which were performed within the framework of the theoretical results of T. Kiselova and F. Wagener [25]. Although these researches are devoted to the study of the behavior of nonlinear dynamic systems in the economy, the systems considered in them are significantly different from the production and economic system that is the object of our research by the nature of their functioning. It should also be noted that in these papers one-parameter bifurcations of the limit cycle type were analyzed without taking into account the global rearrangement of phase trajectories on the two-parameter plane. Results similar to ours were obtained in the research of L. Cheng and L. Zhang [26], where when determining changes in population size in the "predator-prey" model, the possibility of the existence of a different bifurcation structure in the plane of parameters depending on the value of the Lyapunov quantity was revealed. Similarly, as it was done in the mentioned research on the example of the "predator-prey" system, we considered the possibility of existence of Bogdanov-Takens bifurcation for the economic system of nonlinear dynamics, when both coefficients of the characteristic equation can change signs. Just this possibility is a prerequisite for the appearance of the "double zero" bifurcation, that is, the Bogdanov-Takens bifurcation.

It should be noted that knowing the location of bifurcation points and the type of bifurcation that is realized at this point is of great importance, as it marks the transition from one mode of dynamics to another. The very fact of the presence of two positions of equilibrium, which we discovered in the process of qualitative analysis of the production and economic system, leads to a radical restructuring of the understanding of the dynamic behavior of the economic system. However, if the level of conditionally constant costs reaches a critical value, then a catastrophic loss of stability occurs in the system, and such a loss for this cycle is irreversible. Such phenomena should be considered dangerous modes of functioning of the production and economic system, and they are obviously associated with sharp jump-like imbalances, with exterminatory market failures and, apparently, they can be explained only within the framework of the analysis of nonequilibrium systems. The most significant in this sense is the existence of a periodic regime at extremely low frequencies and, accordingly, very long periods of oscillations. This testifies to the theoretical possibility of the appearance of ultra-long waves in the evolution of the economic object under study. In other words, it can be argued that there are so-called "turning points" that change the direction of economic development. Similar to the long waves of the economic conjuncture proposed by Kondratiev, such points are distant from each other by large time intervals. But they have an even more greater distance, that is, such waves can be considered as super long.

Although the model of the production and economic system proposed by us can be considered as a sufficiently simplified formalization of only some qualitative and quantitative characteristics of the functioning of an economic and economic object, but such a formalization made it possible to obtain general information about the development trajectories of this system and its stationary states with the identification of the most significant critical modes functioning. The application of the proposed model for forecasting the evolution of the production and economic system gives the best results in the short- and medium-term time intervals, but in the long-term the model additionally needs to determine the limits of the quantitative forecast, if the required horizon of the quantitative forecast (numerical data) much exceeds the duration of the industrial cycle.

CONCLUSIONS

The resulting mathematical model allows describing the behavior of the production and economic system at a sufficiently high level of generalization, i.e. it gives an idea of the existence of certain phase trajectories of development. This model can be used for any integral economic unit for which an independent closed cycle of reproduction is implemented. Due to this formalization, general information was obtained regarding the trajectories of the system's development and its stationary states, with the identification of the most significant critical modes of functioning, such as the breakdown of the steady state. The study of bifurcations that can arise in the system showed that it is a bifurcation of the local "saddle-node" type, the appearance of which leads to the birth of a cycle, and also a global bifurcation with the presence of a separatrix cycle, which distinguishes periodic and aperiodic types of system development. Using this model, a decision-maker can maintain the stability of a complex system in which feedback takes place. A further step can be simulation modeling using this model to obtain quantitative characteristics of the stability of the system states by assigning specific values to the system parameters.

A feature of the mathematical model of economic dynamics presented by the authors in this paper is the assumption that the evolution of the production and economic system is continuous over time. Since the characteristics of the system are measured at fixed points in time, the subject of further research may be the construction of a similar model for determining processes described in discrete time. The construction of such models requires the use of another mathematical apparatus, namely, equations in finite differences.

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Нелінійна математична модель динаміки виробничо-господарських об’єктів

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Анотація. Особа, що приймає рішення щодо керування виробничо-господарським об’єктом, відчуває потребу в інструментах для вирішення численних проблем, що виникають у процесі функціонування цього об’єкта як економічної системи в умовах взаємодії з навколишнім середовищем. Метою даної роботи є побудова аналітичної моделі виробничо-економічної системи, яка б дозволяла досліджувати структурні зміни, що можуть відбуватися в процесі функціонування господарських об’єктів із замкнутим циклом виробничої діяльності і які визначають можливі шляхи еволюції відкритої економічної системи у часі (траєкторій розвитку). Для побудови моделі було застосовано методологію нелінійної динаміки й економічної синергетики. У роботі запропоновано математичну модель виробничо-економічної системи з невеликою кількістю фазових змінних, що мають ринкову інтерпретацію, та визначені ендогенні та екзогенні параметри, які характеризують стан системи і напрям її розвитку. Модель містить систему двох звичайних диференціальних рівнянь з квадратичною нелінійністю. Така формалізація дозволила отримати загальну інформацію щодо траєкторій розвитку цієї системи і її стаціонарних станів з виявленням найбільш значущих критичних режимів функціонування. Якісний аналіз на основі цієї моделі показав, що нелінійність призводить до неєдиності станів рівноваги та до існування як стійких, так і нестійких траєкторій еволюції досліджуваної економічної системи. Ця модель може бути використана для керування будь-якою цілісною господарською одиницею, в якій забезпечується самостійний замкнутий цикл відтворення

Ключові слова: економічна динаміка, математична модель у неперервному часу, нелінійна динаміка, синергізм, фазові траєкторії еволюції, стійкість точок рівноваги, біфуркація