"S&P 500 Consumer Staples ETF Components". *Barchart*. https://www.barchart.com/stocks/indices/sp-sector/consumer-staples?viewName=main

Schubert, E., and Rousseeuw, P. J. "Fast and eager k-medoids clustering: O(k) runtime improvement of the PAM, CLARA, and CLARANS algorithms". *Information Systems*, vol. 101 (2021). DOI: https://doi.org/10.1016/j.is.2021.101804

"The Global Industry Classification Standard". https://www.msci.com/our-solutions/indexes/gics

Tan, W. et al. "Method towards reconstructing collaborative business processes with cloud services using evolutionary deep Q-learning". *Journal of Industrial Information Integration*, vol. 21, no. 2021. DOI: https://doi.org/10.1016/j.jii.2020.100189

UDC 330.322.7
JEL: C62; C67
DOI: https://doi.org/10.32983/2222-4459-2024-9-161-170

STABILITY OF EQUILIBRIUM AND BIFURCATION BEHAVIOR OF THE PRODUCTION AND ECONOMIC SYSTEM

©2024 **MALYARETS L. M., VORONIN A. V., LEBEDEVA I. L.**

UDC 330.322.7
JEL: C62; C67

Malyarets L. M., Voronin A. V., Lebedeva I. L. Stability of Equilibrium and Bifurcation Behavior of the Production and Economic System

This study considers some key problems of analysis of nonlinear dynamic systems on the example of production and economic objects. The construction of a mathematical model of systems, functioning in the market environment, is aimed at qualitative forecasting (along the development trajectory) of the behavioral properties of such a system. The conceptual orientation of the study involves the analysis of structural instability of equilibrium states within the framework of the proposed model by the presence of characteristic combinations of the most important economic parameters that have a significant impact on the static and dynamic characteristics of the production and economic system. Critical modes of functioning of the object have been identified and its stability area has been built in three-dimensional space depending on significant parameters. An example of such dynamic modes, which were revealed in the process of analysis using the proposed model, is an unstable boundary cycle that provokes the so-called «hard» mechanism of excitation of self-oscillations around the equilibrium state of the «focus» type. Also in this system, the global bifurcation of the saddle joint in the presence of a loop of the «saddle» separatrix around the state of equilibrium was found and studied. Such modes are rather dangerous, since there are cyclic processes with very long periods, which significantly affects the accuracy of predicting the behavior of the object under study.

Keywords: *«supply and demand» system, area of system stability, structural instability, critical mode of functioning, bifurcation, forecasting by phase trajectories.*
Fig.: *2.* ***Formulae:*** *40.* ***Bibl.:*** *20.*

Malyarets Lyudmyla M. *– D. Sc. (Economics), Professor, Head of the Department of Economic and Mathematical Modeling, Simon Kuznets Kharkiv National University of Economics (9a Nauky Ave., Kharkiv, 61166, Ukraine)*
E-mail: *malyarets@ukr.net*
ORCID: *https://orcid.org/0000-0002-1684-9805*
Researcher ID: *https://www.webofscience.com/wos/author/record/T-9858-2018*
Scopus Author ID: *https://www.scopus.com/authid/detail.uri?authorId=57189248374*

Voronin Anatolii V. *– PhD (Engineering), Associate Professor, Associate Professor of the Department of Economic and Mathematical Modeling, Simon Kuznets Kharkiv National University of Economics (9a Nauky Ave., Kharkiv, 61166, Ukraine)*
E-mail: *voronin61@ ukr.net*
ORCID: *https://orcid.org/0000-0003-2570-0508*
Scopus Author ID: *https://www.scopus.com/authid/detail.uri?authorId=58677148800*

Lebedeva Irina L. *– PhD (Physics and Mathematics), Associate Professor, Associate Professor of the Department of Economic and Mathematical Modeling, Simon Kuznets Kharkiv National University of Economics (9a Nauky Ave., Kharkiv, 61166, Ukraine)*
E-mail: *irina.lebedeva@hneu.net*
ORCID: *https://orcid.org/0000-0002-0381-649X*
Scopus Author ID: *https://www.scopus.com/authid/detail.uri?authorId=57196850420*

УДК 330.322.7
JEL: C62; C67

Малярець Л. М., Воронін А. В., Лєбєдєва І. Л. Стійкість рівноваги та біфуркаційна поведінка виробничо-економічної системи

У даному дослідженні розглянуті деякі ключові проблеми аналізу нелінійних динамічних систем на прикладі виробничо-економічних об'єктів. Побудова математичної моделі систем, функціонування яких відбувається в ринковому середовищі, спрямована на якісне прогнозування (за траєкторією розвитку) поведінкових властивостей такої системи. Концептуальна спрямованість дослідження передбачає аналіз структурної нестабільності рівноважних станів у межах запропонованої моделі за наявністю характерних сполучень найбільш важливих економічних параметрів, які мають суттєвий вплив на статичні та динамічні характеристики виробничо-економічної системи. Виявлено критичні режими функціонування об'єкта та у тривимірному просторі побудовано його область стійкості залежно від значущих параметрів. Прикладом таких динамічних

режимів, що були виявлені в процесі аналізу з використанням запропонованої моделі, є нестійкий граничний цикл, що провокує так званий «жорсткий» механізм збудження автоколивань навколо рівноважного стану типу «фокус». Також у даній системі було знайдено та досліджено глобальну біфуркацію сідлового з'єднання за наявності петлі сепаратриси «сідла» навколо стану рівноваги. Такі режими є достатньо небезпечними, оскільки існують циклічні процеси з дуже великими періодами, що суттєво впливає на точність прогнозування поведінки досліджуваного об'єкта.
Ключові слова: *система «попит – пропозиція», область стійкості системи, структурна нестабільність, критичний режим функціонування, біфуркації, прогнозування за фазовими траєкторіями.*
Рис.: *2.* ***Формул:*** *40.* ***Бібл.:*** *20.*
Малярець Людмила Михайлівна *– доктор економічних наук, професор, завідувачка кафедри економіко-математичного моделювання, Харківський національний економічний університет імені Семена Кузнеця (просп. Науки, 9а, Харків, 61166, Україна)*
E-mail: *malyarets@ukr.net*
ORCID: *https://orcid.org/0000-0002-1684-9805*
Researcher ID: *https://www.webofscience.com/wos/author/record/T-9858-2018*
Scopus Author ID: *https://www.scopus.com/authid/detail.uri?authorId=57189248374*
Воронін Анатолій Віталійович *– кандидат технічних наук, доцент, доцент кафедри економіко-математичного моделювання, Харківський національний економічний університет імені Семена Кузнеця (просп. Науки, 9а, Харків, 61166, Україна)*
E-mail: *voronin61@ ukr.net*
ORCID: *https://orcid.org/0000-0003-2570-0508*
Scopus Author ID: *https://www.scopus.com/authid/detail.uri?authorId=58677148800*
Лебєдєва Ірина Леонідівна *– кандидат фізико-математичних наук, доцент, доцент кафедри економіко-математичного моделювання, Харківський національний економічний університет імені Семена Кузнеця (просп. Науки, 9а, Харків, 61166, Україна)*
E-mail: *irina.lebedeva@hneu.net*
ORCID: *https://orcid.org/0000-0002-0381-649X*
Scopus Author ID: *https://www.scopus.com/authid/detail.uri?authorId=57196850420*

The functioning of the production and economic system is a complex process wherein various economic indicators (production, consumption, investments, resources, etc.) interact with each other both directly and indirectly. To model the stability of such a system, it is expedient to use the mathematical apparatus of nonlinear dynamics, enabling to study the behavior of a relevant system and identify the limits of its stability. Building mathematical models of complex systems to understand the relationships that operate within the system and predicting changes in its states under the influence of external factors allow to make informed decisions necessary for managing the development of the production and economic system.

The principles of constructing models of complex systems of different nature and the mathematical apparatus, used for this purpose in different fields of knowledge, have much in common [1–3 and others]. This led to the formation of a separate field of knowledge, called system dynamics. System dynamics is considered as an aggregate of methods for building mathematical models that explore the structure of cause-and-effect relationships between the elements of the system, with attention to the nonlinearity of these relationships and the presence of a time lag in their implementation. The main result of modeling is the forecasting of the trajectories of the development of the state of the system in time for the period for which the simulation is carried out. System dynamics also involves the study of both positive and negative feedback loops that may exist in the relevant system. Positive feedback leads to an increase in the effects of changes in factors, and the system can, virtually, go into disarray. On the contrary, the presence of negative (or self-correcting) loops causes resistance to change and stabilizes the state of the system. In nonlinear systems, so-called attractors can be featured [4; 5 and others], i. e.: a point or set of points in phase space to which the state of the system moves in time along a certain phase trajectory. Thus, in the «supply and demand» system, the equilibrium of these factors can be considered as an attractor, but it is not always linearly stable [6].

The development of a system in time can occur either through evolution and revolution, or catastrophe, when there is an abrupt change of state. Mathematical models of nonlinear dynamics play a special role in the study of bifurcations, when in response to a seemingly insignificant change in external influence, a significant change in the parameters of the system develops, as a result of which the behavior of the system qualitatively changes and can even move from an ordered state to a state of chaos. Bifurcation analysis makes it possible to determine the conditions of stability and adaptability of production and economic systems, and the determination of critical points at which the behavior of the system changes and becomes either cyclical or chaotic, helps to prevent crisis phenomena. Although research in this direction has been carried out for a long time [7; 8 and others], the problem remains relevant [9–12 and others]. It should be noted that the use of nonlinear approaches in modeling production systems allows for a more accurate description of real economic dynamics, which often cannot be explained within terms of linear models.

When constructing mathematical models of the dynamics of production systems, it is expedient to use nonlinear differential equations or difference equations, which allows taking into account such effects as the delay in the time of the system's response to changes in factors (lag), the influence of the elasticity of external and internal factors, the presence of an inverse relationship, as well as the possibility of market saturation.

The search for new opportunities and mechanisms that can increase efficiency and ensure the rationality of the structure of activities of economic objects is associated with the allocation of strategic parameters that determine the competitiveness of enterprises through the improvement of commodity policy, the spread of the latest technologies, and the introduction of effective managerial decisions. When constructing a mathematical model of such activity of the production and economic system, a very important stage is the selection of numerical indicators or parameters (their quantity being not a large number) that characterize the internal state of the system and change in time under the influence of various external factors.

Since the implementation of each production process requires certain time expenditures due to the relevant technological constraints, when creating a model of the economic and production system, it is necessary to take into account the dynamic nature of economic factors and criteria that determine the state of this system. The *aim* of this study is to build a mathematical model of the production and economic system, which would allow to carry out qualitative forecasting of changes in the states of a relevant system, that is, to determine the possible states of the system along the trajectories of its development in accordance with the change in the studied characteristics for the short and medium term. This forecasting term means that the forecast horizon is determined based on the duration of the industrial and technological cycle. Since the implementation of each production process requires a certain amount of time, which in turn causes appropriate technological limitations, when modeling such a system, it is necessary to take into account the dynamic nature of economic factors and criteria that determine the state of this system.

In order to build a model of the dynamics of the economic and production system, the mathematical apparatus of the theory of differential and integral-differential equations was used, allowing to describe the change in the characteristic parameters of the system in continuous time. When building the mathematical model, the presence of loops of both positive and negative feedback was taken into account, which determined the nonlinear nature of the dynamics of the system. Such a model allows for a qualitative analysis of the development of the system, that is, along phase trajectories. The use of this approach makes it possible to determine the boundary values of the characteristic parameters, during the transition through which the system loses stability, as well as to determine the type of bifurcations that may occur in this case.

Traditionally, the production system uses its profit for the needs of industrial activity and for the benefit of savings:

$$E = I + S, \qquad (1)$$

where E is the amount of profit; I – the volume of industrial investment; S – the amount of savings.

If the investment component of profit is focused on increasing the efficiency of the production process and improving technological processes, then savings are made for the purpose of further use of financial opportunities for the formation of reserve funds, conducting the necessary research and design work. Thus, it can be stated that investments and savings have a corresponding impact on the regulation of technological and organizational changes in the structure of production.

Let us use the standard statement that the market demand for the enterprise's products is determined by a linearly decreasing function of the unit price:

$$D = d_0 - d_1 p, \qquad (2)$$

where $D = D(p)$ is market demand function; $p = p(t)$ – unit price depending on the time variable t; d_0, d_1 – parameters of the demand function, respectively d_0 – the volume of «ideal» demand, d_1 – the absolute value of elastic demand if the price is p. In the relation (2), it is assumed that the price of a unit of production depends on time. It should be noted that when constructing a mathematical model of objects, the functioning of which takes place in the market environment, it is necessary to consider that the interaction of the volume of production output and the price per unit of goods is a process that is prolonged in time.

Note that all products appear on the market without the formation of stocks of goods. The dynamic balance of supply and demand of goods is given in the form of:

$$D(p(t)) = \int_0^t K(t, \tau) \cdot y(\tau)\, d\tau, \qquad (3)$$

where $y = y(t)$ is the volume of output of goods, which is essentially supply; $K(t, \tau)$ – some weight function.

The integral relation (3) means that the demand at the relevant point in time t depends on all previous values of the volume of supply $y(t)$, $0 \le \tau \le t$. It is obvious that the structure of the dynamic process is completely determined by the form of the weight function $K(t, \tau)$, which has the meaning of the so-called «descending dynamic memory». If we assume that

$$K(t, \tau) = \beta \cdot e^{-\beta(t-\tau)}, \quad (4)$$

then (3) takes the following form:

$$d_0 - d_1 \cdot p(t) = \beta \int_0^t e^{-\beta(t-\tau)} \cdot y(\tau)\, d\tau. \quad (5)$$

Note that kernel formalization is a way of describing the «descending» memory of the previous values of the variable under study.

After differentiating the relation (5) by the variable t, we obtain a first-order differential equation:

$$\frac{dp}{dt} = \beta\left(b - \frac{y}{a} - p\right), \quad (6)$$

where $a = d_1$; $b = \frac{d_0}{d_1}$ is price at zero demand; β – a characteristic time constant, which can be considered the coefficient of market adaptation.

Profit from products sold is determined by the formula:

$$E = (p - c)y - c_0, \quad (7)$$

where c_0 is conditionally stable expenses; c – conditionally variable expenses, which is measured as prices.

The amount of saving is proportional to the amount of profit:

$$S = s \cdot E, \quad (8)$$

where s is the boundary propensity (multiplier) of savings.

The corresponding expression for investments is as follows:

$$I = (1 - s)E. \quad (9)$$

The volume of industrial output is carried out at the expense of accumulated investment resources:

$$y(t) = \int_0^t \gamma \cdot I(\tau) \cdot dr, \quad (10)$$

where γ is the parameter that determines the boundary expenses on increasing the volume of production per unit.

After differentiating the equation (10) taking into account the specific form of $I(t)$ we obtain the following differential equation:

$$\frac{dy}{dt} = \alpha((p - c) \cdot y - c_0), \quad (11)$$

where $\alpha = \gamma(1 - s) > 0$.

It is obvious that differential equations (6) and (11) should be combined into a common system consisting of two differential equations, and this system will determine the dynamics of the interaction between the unit price $p(t)$ and the volume of output $y = y(t)$:

$$\begin{cases} \frac{dy}{dt} = \alpha((p - c) \cdot y - c_0); \\ \frac{dp}{dt} = \beta\left(b - \frac{y}{a} - p\right). \end{cases} \quad (12)$$

The analysis of the evolutionary properties of a system (12) must begin with the search for special solutions, if there are conditions for their existence. A system of algebraic equations for finding special solutions of p^* and y^* has the following form:

$$\begin{cases} (p - c) \cdot y - c_0 = 0; \\ b - \frac{y}{a} - p = 0. \end{cases} \quad (13)$$

After the necessary transformations we get:

$$\begin{cases} (p^2 - (b + c)p + bc + c_0 \cdot a^{-1} = 0; \\ y = a \cdot (b - p). \end{cases} \quad (14)$$

Having solved the system (14), we have the corresponding solutions:

$$p^*_{1,2} = \frac{b + c}{2} \pm \sqrt{\left(\frac{b - c}{2}\right)^2 - \frac{c_0}{a}}, \quad (15)$$

$$y^*_{1,2} = a \cdot \left(\frac{b - c}{2} \mp \sqrt{\left(\frac{b - c}{2}\right)^2 - \frac{c_0}{a}}\right), \; y^*_1 > y^*_2. \quad (16)$$

Analysis of expressions (15) and (16), which define special solutions of the system (12), allows us to formulate the condition for the existence of two different equilibrium points:

$$b > c + 2 \cdot \sqrt[3]{c_0 \cdot a^{-1}}. \quad (17)$$

This means that the price of «zero» demand is always higher than the conditionally variable expenses.

For a further proceeding with studying the properties of the system (12), we introduce a new time scale $t_0 = \alpha \cdot t$ and new variables $\tilde{p} = p - p^*$ and $\tilde{y} = y - y^*$, that determine deviations from their equilibrium states (i. e., from special solutions). Then for the new variables, the system (12) takes the following form:

$$\begin{cases} \frac{d\tilde{y}}{dt_0} = (p^* - c) \cdot \tilde{y} + a(b - p^*) \cdot \tilde{p} + \tilde{y} \cdot \tilde{p}; \\ \frac{d\tilde{p}}{dt_0} = -\frac{\theta}{a}\tilde{y} - \theta \cdot \tilde{p}, \end{cases} \quad (18)$$

where $\theta = \frac{\beta}{\gamma}$.

The matrix of the linear part of the system (18) is as follows:

$$\mathbf{A} = \begin{pmatrix} p^* - c & a(b - p^*) \\ -\frac{\theta}{a} & -\theta \end{pmatrix}.$$

The characteristic equation for finding the eigennumbers and can be written as follows:

$$\lambda^2 - tr\mathbf{A} \cdot \lambda + \det \mathbf{A} = 0, \qquad (19)$$

where $tr\mathbf{A} = p^* - c - \theta$ is the trace of matrix $\mathbf{A}$; $\det \mathbf{A} = \theta(b + c - 2p^*)$ – determinant of the same matrix.

The quadratic equation (19) allows us to formulate unambiguous conditions under which the real parts of the eigennumbers λ_1 and λ_2 will be negative. This coincides with the stability conditions of the corresponding equilibrium state of the system (18):

$$\begin{cases} p^* - c - \theta < 0; \\ b + c - 2p^* > 0. \end{cases} \qquad (20)$$

From the second inequality of the system (20) follows that

$$p^* < \frac{b+c}{2}, \qquad (21)$$

and this automatically ensures the instability of the equilibrium state when $p_2^* = \frac{b+c}{2} + \sqrt{\left(\frac{b-c}{2}\right)^2 - \frac{c_0}{a}}$ is realized. In this case, the determinant of the matrix **A** of the linear part of the system (18) will be negative, with a singular point (y_2^*, p_2^*) that is of the saddle type. Obviously, inequality (21) always holds for a smaller price $p_1^* = \frac{b+c}{2} - \sqrt{\left(\frac{b-c}{2}\right)^2 - \frac{c_0}{a}}$. Then the first of the inequalities of the system (20) turns into the following expression:

$$\theta > \frac{b+c}{2} - \sqrt{\left(\frac{b-c}{2}\right)^2 - \frac{c_0}{a}}. \qquad (22)$$

Fig. 1 shows a graphical representation of the stability region of the equilibrium state (y_1^*, p_1^*), where $z > 0, x > 0, y > 0$.

If we analyze the mathematical content of the expressions for the trace and the determinant of the matrix of the linear part of the system (18), we can state the change in the signs of both quantities depending on the parameters of the system. Therefore, it is logical to assume that the trace and determinant of this matrix are small alternating quantities, i. e.

$$\begin{cases} c + \theta - p^* = \xi_2; \\ \theta(b + c - 2p^*) = \xi_1. \end{cases} \qquad (23)$$

By substituting the formula (15) for the equilibrium price into system (23), we obtain the following expressions:

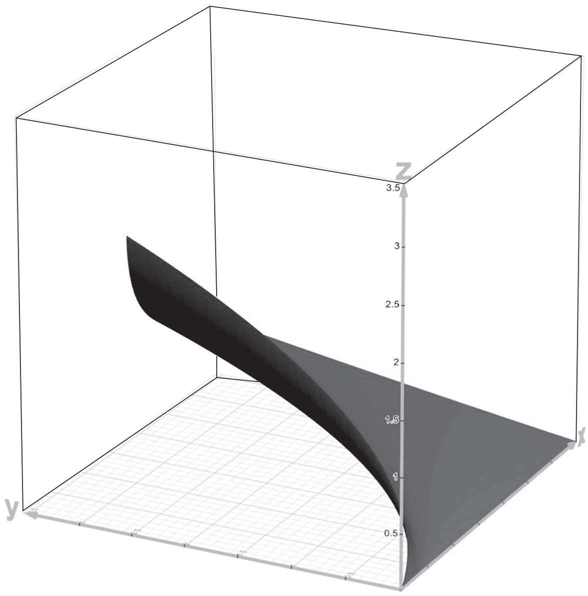

Fig. 1. Lower limit of the stability region of the equilibrium state

$$c = b - 2\theta + 2\xi_2 - \frac{\xi_1}{\theta}; \qquad (24)$$

$$c_0 = a(\theta^2 + \xi_1 - 2\theta \cdot \xi_2). \qquad (25)$$

The relations (24) and (25) demonstrate the expediency of choosing conditionally variable and conditionally stable expenses as bifurcation parameters.

After the selection of c and c_0 respective of the relations (24) and (25) we rewrite the system (18) as follows:

$$\begin{cases} \frac{d\tilde{y}}{dt_0} = (\theta - \xi_2) \cdot \tilde{y} + a\left(\theta - \xi_2 + \frac{\xi_1}{\theta}\right) \cdot \tilde{p} + \tilde{y} \cdot \tilde{p}; \\ \frac{d\tilde{p}}{dt_0} = -\frac{\theta}{a}\tilde{y} - \theta \cdot \tilde{p}. \end{cases} \qquad (26)$$

The linear part of the system (26) has the following characteristic equation:

$$\lambda^2 + \xi_2 \cdot \lambda + \xi_1 = 0. \qquad (27)$$

In the case when $\xi_1 = \xi_2 = 0$, the solution of equation (27) is a twofold zero, i. e. $\lambda_1 = \lambda_2 = 0$. Thus, it is possible to propose a hypothesis about the existence of a Bogdanov – Takens bifurcation [13; 14] in the nonlinear system of differential equations (27).

For a more substantive study of the properties of the bifurcation of codimension-two, the so-called «double-zero bifurcation», it is necessary to find a corresponding normal form. Using the linear substitution of the variables $\tilde{y} = -\theta x_1 - x_2$, $\tilde{p} = \frac{\theta}{a}x_1$ the system (26) can be written down as follows:

$$\begin{cases} \dfrac{dx_1}{dt_0} = x_2; \\ \dfrac{dx_2}{dt_0} = -\xi_1 x_1 - \xi_2 x_2 + \dfrac{\theta}{a} x_1 x_2 + \dfrac{\theta^2}{a} x_1^2. \end{cases} \quad \textbf{(28)}$$

In order to obtain an explicit form of the normal form of the bifurcation under study, it is necessary to make one more substitution of variables: $x_1 = a\left(y_1 + \dfrac{\xi_1}{2\theta^2}\right)$, $x_2 = a\theta y_2$, $t_0 = \dfrac{\tau}{\theta}$. After the necessary transformation, we obtain:

$$\begin{cases} \dfrac{dy_1}{d\tau} = y_2; \\ \dfrac{dy_2}{d\tau} = \eta_1 + \eta_2 y_2 + y_1 y_2 + y_1^2. \end{cases} \quad \textbf{(29)}$$

where $\eta_1 = -\dfrac{\xi_1^2}{4\theta^4}$, $\eta_2 = \dfrac{\xi_1}{2\theta^2} - \dfrac{\xi_2}{\theta}$.

The system (29) has two equilibrium points (singular points):

$$(y_1^*; y_2^*) = (\pm\sqrt{-\eta_1}; 0), \quad \textbf{(30)}$$

which have always existed, because $\eta_1 < 0$.

The system (29) linearized in the vicinity of these points has a corresponding matrix:

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 \\ \pm\sqrt{-\eta_1} & \eta_2 \pm \sqrt{-\eta_1} \end{pmatrix}.$$

Since the matrix $\mathbf{A}_0$ has an explicit form, we can conclude that the point $(+\sqrt{-\eta_1}; 0)$ is always stable, and additional studies need to be carried out with respect to the point $(-\sqrt{-\eta_1}; 0)$.

With regard to the immobility of the point $(-\sqrt{-\eta_1}; 0)$ it can be noted that it is unstable under the condition of the inequality $\eta_2 > \sqrt{-\eta_1}$ and, conversely, is stable if $\eta_2 < \sqrt{-\eta_1}$. Consequently, there is a shift from unstable focus to steady focus on the line $\eta_2 = \sqrt{-\eta_1}$. Given that the system (29) is a nonlinear system of two differential equations, it is necessary to check whether the Hopf bifurcation is present in the system under study and to determine its type. We reason that theoretically there can be both stable and unstable modes of self-oscillation.

It can be shown that in the system (26) there is another interesting saddle-node bifurcation on the straight line $\eta_1 = 0, \eta_2 \neq 0$. This phenomenon occurs when the determinant of the matrix $\mathbf{A}_0$ changes its sign. If this sign is positive, then the fixed point $(-\sqrt{-\eta_1}; 0)$ is a stable node, and in the opposite case, when the determinant has a negative sign, then the point under study acquires a saddle type of equilibrium. The loss of stability of a fixed point is accompanied by a fold type catastrophe.

In order to study the stability of the Hopf bifurcation, it is necessary to perform two substitutions of variables, the first of which will bring the vector field (29) to the standard form for this bifurcation. Suppose that $\tilde{y}_1 = y_1 + \sqrt{-\eta_1}$ and $\tilde{y}_1 = y_2$. Considering that $\eta_2 = \sqrt{-\eta_1}$, the system (29) takes the form:

$$\begin{cases} \dfrac{d\tilde{y}_1}{d\tau} = \tilde{y}_2; \\ \dfrac{d\tilde{y}_2}{d\tau} = -2\sqrt{-\eta_1} \cdot \tilde{y}_1 + \tilde{y}_1 \cdot \tilde{y}_2 + \tilde{y}_1^2. \end{cases} \quad \textbf{(31)}$$

Now let us do the scaling conversion $\tilde{y}_1 = u_2$, $\tilde{y}_2 = \sqrt{2\sqrt{-\eta_1}} \cdot u_1$, the matrix of which is formed by means of eigenvectors corresponding to the eigenvalues of $\lambda_{1,2} = \pm i\sqrt{2\sqrt{-\eta_1}}$ $(i^2 = 1)$. Thus, we obtain the system, the linear part of which is provided in a standard form:

$$\begin{cases} \dfrac{du_1}{d\tau} = -\sqrt{2\sqrt{-\eta_1}} \cdot u_2 + u_1 u_2 + \dfrac{1}{\sqrt{2\sqrt{-\eta_1}}} \cdot u_2^2; \\ \dfrac{du_2}{d\tau} = \sqrt{2\sqrt{-\eta_1}} \cdot u_1. \end{cases} \quad \textbf{(32)}$$

Let us introduce a complex variable $z = u_1 + i \cdot u_2$ and its conjugate variable $\bar{z} = u_1 - i \cdot u_2$. Then the system (32) is transformed into a first-order differential equation:

$$\frac{dz}{d\tau} = i\sqrt{2\sqrt{-\eta_1}} \cdot z + g_{20}\frac{z^2}{2} + g_{11} z \cdot \bar{z} + g_{02}\frac{\bar{z}^2}{2}, \quad \textbf{(33)}$$

where $g_{20} = \dfrac{1}{2\omega} - \dfrac{1}{2}i$; $g_{11} = \dfrac{1}{2\omega}$;

$g_{02} = \dfrac{1}{2\omega} + \dfrac{1}{2}i$; $\omega = \sqrt{2\sqrt{-\eta_1}}$.

We proceed to find the first Liapunov value:

$$l_1 = \mathrm{Re}\left(\frac{i}{\sqrt{2\sqrt{-\eta_1}}} g_{20} \cdot g_{11}\right) = \frac{1}{16\sqrt{-\eta_1}} > 0. \quad \textbf{(34)}$$

From the relation (34) we obtained that the first Liapunov quantity is positive. This means that the Hopf bifurcation is subcritical, i. e. there is an aggregate of unstable periodic orbits around the stable focus for the parameter values η_2, for which the inequality: $\eta_2 < \sqrt{-\eta_1}$ holds and, at the same time, these values of the parameter differ little from the value of $\sqrt{-\eta_1}$. The loss of focus stability is abrupt in nature with a corresponding catastrophic mode of excitation of self-oscillations.

The presence in the system (29) of two fundamentally different types of bifurcations, such as the «saddle-node» bifurcation and the bifurcation of appearance or disappearance of the boundary cycle, allows the occurrence of a global bifurcation with the formation of a saddle separatrix loop. To study such a bifurcation, we need to make another substitution of variables in the system (29):

$$y_1 = \delta^2 \nu_1;\ y_2 = \delta^2 \nu_2;\ \eta_1 = \delta^4 \mu_1;\ \eta_2 = \delta^2 \mu_2;\ \tau_0 = \varepsilon\tau.$$

By substituting these variables into the system (29), we get:

$$\begin{cases} \dfrac{d\nu_1}{d\tau_0} = \nu_2; \\ \dfrac{d\nu_2}{d\tau_0} = \mu_1 + \delta\mu_2\nu_2 + \delta\nu_1\nu_2 + \nu_1^2. \end{cases} \quad \textbf{(35)}$$

If we assume that δ = 0, then the system (35) becomes Hamiltonian [15] with the existence of the so-called first integral:

$$\begin{cases} \dfrac{d\nu_1}{d\tau_0} = \nu_2; \\ \dfrac{d\nu_2}{d\tau_0} = \mu_1 + \nu_1^2. \end{cases} \quad \textbf{(36)}$$

On the phase plane, the system (36) is reduced to one equation:

$$\frac{d\nu_2}{d\nu_1} = \frac{\mu_1 + \nu_1^2}{\nu_2}$$

with the appropriate solution:

$$H(\nu_1, \nu_2) = \frac{\nu_2^2}{2} - \mu_1\nu_1 - \frac{\nu_1^3}{3}. \quad \textbf{(37)}$$

The function (37) is called the Hamiltonian, or the first integral of the system (36). The phase picture of the system (36) is shown in *Fig. 2*.

The next step in our research is to find the parameter μ_2 at $\varepsilon \to 0$, for which there is a saddle connection, that is, the point of intersection of two symmetrical loops on the «saddle» graph (see Fig. 2). This problem can be solved using the Melnikov method [16–18]. The solution corresponding to the function (37) at the base point $\nu_1(0) = -2;\ \nu_2(0) = 0$ is given by the formula:

$$\begin{cases} \nu_1(\tau_0) = 1 - 3\,\mathrm{sech}^2\left(\dfrac{\tau_0}{\sqrt{2}}\right); \\ \nu_2(\tau_0) = 3\sqrt{2}\,\mathrm{sech}^2\left(\dfrac{\tau_0}{\sqrt{2}}\right) \cdot th\left(\dfrac{\tau_0}{\sqrt{2}}\right). \end{cases} \quad \textbf{(38)}$$

In this case, the Melnikov function is stationary and is computed by the formula as an improper integral:

$$M(\mu_2) = \int_{-\infty}^{+\infty} \nu_2(t)(\mu_2 \cdot \nu_2(t) + \nu_1(t) \cdot \nu_2(t))\,dt.$$

By integrating and equating the function $M(\mu_2)$, we get:

$$M(\mu_2) = 0 \to \mu_2 = \frac{5}{7}. \quad \textbf{(39)}$$

With the help of the relation (39), we get the relationship between the parameters η_1 and η_2:

$$\eta_1 = -\frac{49}{25}\eta_2^2,\ \eta_2 \ge 0. \quad \textbf{(40)}$$

The semiparabola (40) is an approximate line on which the global bifurcation of the saddle connection is realized. It is also important to note that the trace of the linearization matrix of the «saddle» value is positive: $\eta_2 + \sqrt{-\eta_1} = \frac{12}{5}\eta_2 > 0$, which gives reason for stating that the homoclinic orbit is the «alpha» boundary set [19; 20] for close points.

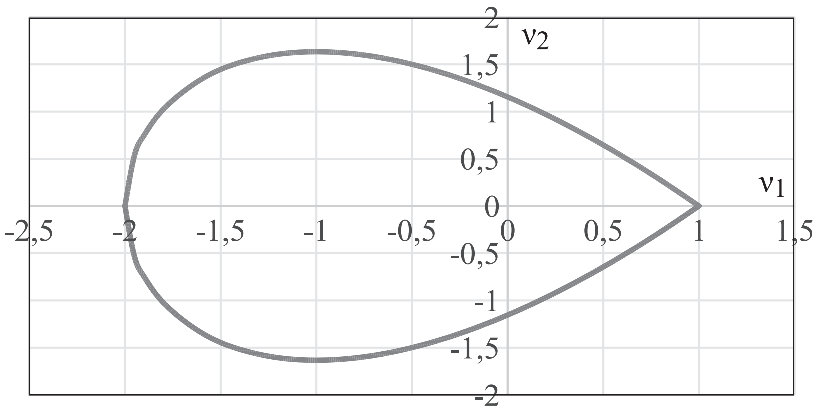

Fig. 2. Phase pattern of the Hamiltonian system for $\mu_1 = -1$

The above analysis of the structural stability of the production and economic system (12) concerns the situation when both equilibrium states are at a small distance from each other. If, for example, the parameter c_0, which reflects the level of conditionally stable expenses, in the process of growth reaches its critical level $c_0^* = \frac{a(b-c)^2}{4}$, then both equilibrium states merge into one and then disappear. At this, the «fold catastrophe» occurs. In the case when the dynamic parameter θ that has the price measurability, approaches the characteristic value $\theta^* = \frac{b-c}{2}$, which can be interpreted as half of the difference between the price of «zero» demand and conditionally variable expenses, an unstable boundary cycle with a «hard» mode of excitation of the self-oscillation process is born in the basic dynamic system (12) with a complex focus. The fact that the cycle loses stability is irreversible in this case.

CONCLUSIONS

Thus, the analysis of the proposed mathematical model of nonlinear dynamics has shown that in the process of development of the economic and production system, phenomena may arise that should be considered dangerous modes of functioning of this system, since they are associated with sharp abrupt disturbances of equilibrium and with destructive «market failures». The above described bifurcation modes and the effects of the system's transition to a state of chaos can be given a meaningful interpretation only from the standpoint of non-equilibrium or dynamic analysis.

As a prospect for further study of the evolution of production systems, it is expedient to consider the dynamic system of the third order, which is associated with the emergence of a new independent variable, for example, the volume of savings. Then the dynamic properties of such a system will be much more complex. This complication is associated with such effects as the appearance of the invariant tori with two different oscillation frequencies, chaotic irregular trajectory, and so on. It is also advisable to consider different ways of formalizing the kernel, since the system is capable to «remember» its previous state and thus the values of the variable under study depend on how long and in what way this remembering accumulates.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest in connection with the current research, including financial, personal, authorial, or any other that could affect the research and results presented in this article. ■

BIBLIOGRAPHY

1. Lu K., Chen Y., Hou L. Bifurcation characteristics analysis of a class of nonlinear dynamical systems based on singularity theory. *Applied Mathematics and Mechanics* (English Edition). 2017. Vol. 38. Iss. 9. P. 1233–1246. DOI: https://doi.org/10.1007/s10483-017-2234-8
2. ElFadily S., Kaddar A., Najib K. Direction and stability of Hopf bifurcation in a delayed Solow model with labor demand. *International Journal of Differential Equations*. 2019. Art. 7609828. DOI: https://doi.org/10.1155/2019/7609828
3. Balthazar J. M., Gonçalves P. B., Tusset A. M. et al. Mathematical modelling, nonlinear dynamics, bifurcation, synchronization and control of mechanisms driven by power supply. *The European Physical Journal Special Topics*. 2021. Vol. 230. P. 3433–3438. DOI: https://doi.org/10.1140/epjs/s11734-021-00300-z
4. Carvalho J. M., Rodrigues A. A. Strange attractors in a dynamical system inspired by a seasonally forced SIR model. *Physica D: Nonlinear Phenomena*. 2022. Vol. 434. Art. 133268. DOI: https://doi.org/10.1016/j.physd.2022.133268
5. Dutta A., Harshith J., Ramamoorthy A., Lakshmanan K. Attractor inspired deep learning for modelling chaotic systems. *Human-Centric Intelligent Systems*. 2023. Vol. 3. P. 461–472. DOI: https://doi.org/10.1007/s44230-023-00045-z
6. Sun M., Tian L., Fu Y. An energy resources demand-supply system and its dynamical analysis. *Chaos, Solitons & Fractals*. 2007. Vol. 32. Iss. 1. P. 168–180. DOI: https://doi.org/10.1016/j.chaos.2005.10.085
7. Palma A., Lefèvre C. Bifurcation and behavior of complex systems. *Applied Mathematics and Computation*. 1984. Vol. 14. Iss. 1. P. 77–95. DOI: https://doi.org/10.1016/0096-3003(84)90047-X
8. Панухник О. В., Маркович І. Б. Теоретичні засади біфуркаційності економічних систем в контексті трансформацій міжсуб'єктних зв'язків. *Актуальні проблеми економіки*. 2015. № 4. С. 51–57. URL: https://elartu.tntu.edu.ua/bitstream/123456789/5675/1/51-57.pdf
9. Лазарева М. В. Поведінка проектно-орієнтованої системи в точці біфуркації. *Управління розвитком складних систем*. 2015. Вип. 24. С. 36–41. URL: https://urss.knuba.edu.ua/files/zbirnyk-24/36-41.pdf
10. Voronin A., Gunko O. Methods of Comparative Statics and Dynamics in the Theory of Economic Cycles. *Сучасний стан наукових досліджень та технологій в промисловості*. 2021. № 2. С. 46–53. DOI: https://doi.org/10.30837/ITSSI.2021.16.046
11. Arbulu R. J., Zabelle T. Bifurcation of demand and supply (Schedules vs. Production Systems). *1st Annual Technical Conference*. 2023. Presentations. URL: https://projectproduction.org/technical-conference/bifurcation-of-demand-and-supply-schedules-vs-production-systems/

12. Otalvaro R. G., Watson B. C. A Framework to Use Bie furcation Analysis for Insight into Complex Systems Resilience. *INCOSE International Symposium* (Dublin, Ireland. July 2–6). 2024. Vol. 34. Iss. 1. P. 2384–2399. DOI: https://doi.org/10.1002/iis2.13276
13. Воронин А. В. Циклы в задачах нелинейной макроэкономики. Харьков : ИД «ИНЖЭК», 2006. 136 с.
14. Malyarets L. M., Voronin A. V., Lebedeva I. L., Lebediev S. S. Innovation Development of an Enterprise: Modeling Dynamics. *Бізнес Інформ*. 2023. № 10. С. 162–174. DOI: https://doi.org/10.32983/2222-4459-2023-10-162-174
15. Rockafellar R. T. Hamiltonian trajectories and saddle points in mathematical economics. *Control and Cybernetics*. 2009. Vol. 38. No. 4B. P. 1575–1588. URL: http://matwbn.icm.edu.pl/ksiazki/cc/cc38/cc38432.pdf
16. Granados A., Hogan S. J., Seara T. M. The Melnikov Method and Subharmonic Orbits in a Piecewise-Smooth System. *SIAM Journal on Applied Dynamic Systems*. 2012. Vol. 11. Iss. 3. DOI: https://doi.org/10.1137/110850359
17. Yang J., Han M. Some properties of Melnikov functions near a cuspidal loop. *Science China Mathematics*. 2024. Vol. 67. P. 767–786. DOI: https://doi.org/10.1007/s11425-022-2124-7
18. Liu X., Han M. Bifurcation of Limit Cycles by Perturbing Piecewise Hamiltonian Systems. *International Journal of Bifurcation and Chaos*. 2010. Vol. 20. No. 05. P. 1379–1390. DOI: https://doi.org/10.1142/S021812741002654X
19. Balibrea F., Guirao J. L.G., Lampart M. A note on the definition of α-limit Set. *Applied Mathematics & Information Sciences*. 2013. Vol. 7. Iss. 5. P. 1929–1932. DOI: 10.12785/amis/070530
20. Plesa T., Vejchodský T., Erban R. Chemical reaction systems with a homoclinic bifurcation: an inverse problem. *Journal of Mathematical Chemistry*. 2016. Vol. 54. P. 1884–1915. DOI: https://doi.org/10.1007/s10910-016-0656-1

REFERENCES

Arbulu, R. J., and Zabelle, T. "Bifurcation of demand and supply (Schedules vs. Production Systems)". *1st Annual Technical Conference*. 2023. https://projectproduction.org/technical-conference/bifurcation-of-demand-and-supply-schedules-vs-production-systems/

Balibrea, F., Guirao, J. L.G., and Lampart, M. "A note on the definition of α-limit Set". *Applied Mathematics & Information Sciences*, vol. 7, no. 5 (2013): 1929-1932. DOI: 10.12785/amis/070530

Balthazar, J. M. et al. "Mathematical modelling, nonlinear dynamics, bifurcation, synchronization and control of mechanisms driven by power supply". *The European Physical Journal Special Topics*, vol. 230 (2021): 3433-3438. DOI: https://doi.org/10.1140/epjs/s11734-021-00300-z

Dutta, A. et al. "Attractor inspired deep learning for modelling chaotic systems". *Human-Centric Intelligent Systems*, vol. 3 (2023): 461-472. DOI: https://doi.org/10.1007/s44230-023-00045-z

ElFadily, S., Kaddar, A., and Najib, K. "Direction and stability of Hopf bifurcation in a delayed Solow model with labor demand". *International Journal of Differential Equations*, art. 7609828 (2019). DOI: https://doi.org/10.1155/2019/7609828

Granados, A., Hogan, S. J., and Seara, T. M. "The Melnikov Method and Subharmonic Orbits in a Piecewise-Smooth System". *SIAM Journal on Applied Dynamic Systems*, vol. 11, no. 3 (2012). DOI: https://doi.org/10.1137/110850359

Lazareva, M. V. "Povedinka proektno-oriientovanoi systemy v tochtsi bifurkatsii" [The Project-Oriented System Behavior in the Bifurcation Point]. *Upravlinnia rozvytkom skladnykh system*, iss. 24 (2015): 36-41. https://urss.knuba.edu.ua/files/zbirnyk-24/36-41.pdf

Liu, X., and Han, M. "Bifurcation of Limit Cycles by Perturbing Piecewise Hamiltonian Systems". *International Journal of Bifurcation and Chaos*, vol. 20, no. 05 (2010): 1379-1390. DOI: https://doi.org/10.1142/S021812741002654X

Lu, K., Chen, Y., and Hou, L. "Bifurcation characteristics analysis of a class of nonlinear dynamical systems based on singularity theory". *Applied Mathematics and Mechanics* (English Edition), vol. 38, no. 9 (2017): 1233-1246. DOI: https://doi.org/10.1007/s10483-017-2234-8

Malyarets, L. M. et al. "Innovation Development of an Enterprise: Modeling Dynamics". *Biznes Inform*, no. 10 (2023): 162-174. DOI: https://doi.org/10.32983/2222-4459-2023-10-162-174

Otalvaro, R. G., and Watson, B. C. "A Framework to Use Bifurcation Analysis for Insight into Complex Systems Resilience". *INCOSE International Symposium*, vol. 34, no. 1 (2024): 2384-2399. DOI: https://doi.org/10.1002/iis2.13276

Palma, A., and Lefevre, C. "Bifurcation and behavior of complex systems". *Applied Mathematics and Computation*, vol. 14, no. 1 (1984): 77-95. DOI: https://doi.org/10.1016/0096-3003(84)90047-X

Panukhnyk, O. V., and Markovych, I. B. "Teoretychni zasady bifurkatsiinosti ekonomichnykh system v konteksti transformatsii mizhsubiektnykh zviazkiv" [Theoretical Grounds for Economic Systems Bifurcation in the Context of Intersubject Relations Transformation]. *Aktualni problemy ekonomiky*, no. 4 (2015): 51-57. https://elartu.tntu.edu.ua/bitstream/123456789/5675/1/51-57.pdf

Plesa, T., Vejchodsky, T., and Erban, R. "Chemical reaction systems with a homoclinic bifurcation: an inverse problem". *Journal of Mathematical Chemistry*, vol. 54 (2016): 1884-1915. DOI: https://doi.org/10.1007/s10910-016-0656-1

Rockafellar, R. T. "Hamiltonian trajectories and saddle points in mathematical economics". *Control and Cy-

bernetics, vol. 38, no. 4B (2009): 1575-1588. http://matwbn.icm.edu.pl/ksiazki/cc/cc38/cc38432.pdf

Sarvalho, J. M., and Rodrigues, A. A. "Strange attractors in a dynamical system inspired by a seasonally forced SIR model". *Physica D: Nonlinear Phenomena*, art. 133268, vol. 434 (2022).
DOI: https://doi.org/10.1016/j.physd.2022.133268

Sun, M., Tian, L., and Fu, Y. "An energy resources demand-supply system and its dynamical analysis". *Chaos, Solitons & Fractals*, vol. 32, no. 1 (2007): 168-180.
DOI: https://doi.org/10.1016/j.chaos.2005.10.085

Voronin, A. V. *Tsikly v zadachakh nelineynoy makroekonomiki* [Cycles in Problems of Nonlinear Macroeconomics]. Kharkiv: ID «INZhEK», 2006.

Voronin, A., and Gunko, O. "Methods of Comparative Statics and Dynamics in the Theory of Economic Cycles". *Suchasnyi stan naukovykh doslidzhen ta tekhnolohii v promyslovosti*, no. 2 (2021): 46-53.
DOI: https://doi.org/10.30837/ITSSI.2021.16.046

Yang, J., and Han, M. "Some properties of Melnikov functions near a cuspidal loop". *Science China Mathematics*, vol. 67 (2024): 767-786.
DOI: https://doi.org/10.1007/s11425-022-2124-7