MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

SIMON KUZNETS KHARKIV NATIONAL UNIVERSITY OF ECONOMICS

Guidelines
to practical tasks in differential calculus
on the academic discipline
"HIGHER AND APPLIED MATHEMATICS"
for full-time foreign students and students taught in English
of training direction 6.030601 "Management"
of specialization "Business Administration"

Харків. Вид. ХНЕУ ім. С. Кузнеця, 2014

Затверджено на засіданні кафедри вищої математики й економіко-математичних методів.

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The sufficient theoretical material on the academic discipline and typical examples are presented to help students master the material on the theme "Differential Calculus of a Function of One Variable" and apply the obtained knowledge to practice. Individual tasks for self-study work and a list of theoretical questions are given to promote the improvement and extension of students' knowledge on the theme.

Recommended for full-time students of training direction 6.030601 "Management".

Викладено необхідний теоретичний матеріал з навчальної дисципліни та наведено типові приклади, які сприяють найбільш повному засвоєнню матеріалу з теми "Диференціальне числення функції однієї змінної" та застосуванню отриманих знань на практиці. Подано завдання для індивідуальної роботи та перелік теоретичних питань, що сприяють удосконаленню та поглибленню знань студентів з даної теми.

Рекомендовано для студентів напряму підготовки 6.030601 "Менеджмент" денної форми навчання.

Introduction

Differential calculus plays a very important rule in economics in particular in problems concerning the optimum, management and plans. Therefore the deep knowledge in this division in higher and applied mathematics is necessary for modern economists.

In guidelines in brief form only the most principal topics of differential calculus are stated.

The present guidelines are the continuation of the one part where notions of limits and continuity of functions had been regarded. By means of these notions we can introduce the notions of derivative and differential of a function which are one of the most fundamental in mathematics.

Guidelines for Differential Calculus of a Function of One Variable

1. Derivative and Differential

Let's begin with considering one of the problems using the notion of the derivative.

Velocity of rectilinear motion. Let a point move in a straight line which is taken as a number scale and let the law of variation of the coordinate s of the moving point as function of time to be known $s = F(t)$.

During time interval Δt from time moment t to $t + \Delta t$ the coordinate of the point gains the increment

$$\Delta s = F(t + \Delta t) - F(t).$$

If the motion is uniform, that is s is a linear function of t of the form $s = s_0 + v_0 t$ we have $\Delta s = v_0 \Delta t$ and $\frac{\Delta s}{\Delta t} = v_0$ is a constant velocity of the rectilinear motion of the point. But if the motion is nonuniform the ratio $\frac{\Delta s}{\Delta t}$ depends on both t and Δt. It is then called the average (mean) velocity corresponding to time interval from t to $t + \Delta t$. Denoting is v_{av} we can write

$v_{av}=\dfrac{\Delta s}{\Delta t}$. Passing to the limit as $\Delta t\to 0$ we come to the definition of the velocity of rectilinear motion at the given moment that is

$$v=\lim_{\Delta t\to 0} v_{av}=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}=\lim_{\Delta t\to 0}\frac{F(t+\Delta t)-F(t)}{\Delta t}.$$

Now we may give a definition of the derivative of a function.

Definition 1. The limit of the ratio of the increment of a given function $y=f(x)$ to the increment of the independent variable as the latter tends to zero is called the derivative of that function (provided that this limit exists):

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Leibnit'z notation of the derivative is $\dfrac{df(x)}{dx}$ or $\dfrac{dy}{dx}$.

The particular value of the derivative $f'(x)$ at a given point x_0 is usually denoted by $f'(x_0)$ or $y'\big|_{x=x_0}$.

Now we can see the connection between the derivative of the function and the velocity of rectilinear motion at the given moment t.

In general case the derivative of a function can be interpreted as the rate of change of a function $f(x)$ at a given point x that is the limit of the average rate of change of that function in the interval $[x, x+\Delta x]$ as x tends to zero (on condition that this limit exists).

Now let's introduce the notion of left and right-hand derivatives at the point x_0:

$f'_+(x)=\lim\limits_{\Delta x\to 0+0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ is the right-hand derivative;

$f'_-(x)=\lim\limits_{\Delta x\to 0-0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ is the left-hand derivative.

There is such a statement. The function $f(x)$ has the derivative at the point x_0 if its left- and right-hand derivatives coincide.

The notion of differential is closely related to the notion of derivative. Let's consider the notion of differential. Since $\lim\limits_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = f'(x_0)$, this implies $\frac{\Delta f(x_0)}{\Delta x} = f'(x_0) + \varepsilon$, where ε is an infinitesimal as $\Delta x \to 0$. It follows that $\Delta f(x_0) = f'(x_0)\Delta x + \varepsilon \Delta x$, i. e. $\Delta f(x_0) = f'(x_0)\Delta x + o(\Delta x)$, where $o(\Delta x)$ is an infinitesimal of higher order than Δx as $\Delta x \to 0$ and $f'(x_0)$ is a constant. Then the quantity $f'(x_0)\Delta x$ is the linear part of the increment of the function. And now we can present a definition of the differential.

Definition 2. The principal linear part $f'(x_0)\Delta x$ of the increment $\Delta f(x)$ of the function which is proportional Δx is called the differential of the function. The differential of the function is denoted as $df(x)$ or dy if $y = f(x)$.

Thus, the differential and the derivative are connected by the relationship $dy = f'(x)\Delta x$. The increment Δx of the independent variable is called its differential and denoted by dx, where $dx = \Delta x$.

This is coherent with the general definition of the differential since for the function $y = x$ we have

$$dy = dx = x'\Delta x = \Delta x \text{ that is } dx = \Delta x.$$

Thus, the differential of the function is equal to its derivative multiplied by the differential of the independent variable:

$$dy = f'(x)dx.$$

2. Economic Significance of the Derivative

In practice of economic investigations the so-called production functions are widely used for revealing of relationships between the output of produce and resources input, for prognostication of the development growth of industry, for solution of optimum problems and others.

In supposing of differentiability of production function the differential characteristics, connected with the notion of the derivative, gain an important meaning. For instance, if the production function $y = f(x)$ establishes the relation of the output of produce y on resources input x, then $f'(x)$ is called the marginal product, if $y = f(x)$ asserts the relation of the cost of production y on the output of produce x, then $f'(x)$ is called marginal cost.

The characteristic of the relative change of the function increase $y = f(x)$ to be relative small increase of the argument x is called the coefficient of elasticity. The coefficient of elasticity is defined by the formula

$$E = \frac{dy}{y} : \frac{dx}{x} \quad \text{or} \quad E = y' \frac{x}{y}.$$

This coefficient is widely used in investigations of consumers commodity demand in the dependence on prices of the commodities and incomes of population.

The high coefficient of elasticity means the weak power of the consumer's saturation and the low coefficient indicates to the large insistency of this saturation.

Now we proceed to the geometrical meaning of the derivative and the differential.

3. The Geometrical Meaning of the Derivative. The Tangent and the Normal to a Line. The Geometrical Significance of the Differential

Let's consider the graph of the function $f(x)$ and draw a straight line passing though points $M_0(x_0, y_0)$ and $M_1(x_1, y_1)$. It will be called the secant passing through these points (Fig. 1). The points of the secant with coordinates (x, y) satisfy the equality

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \tan \alpha_s,$$

where α is the angle between the secant and the x-axis. Let us rewrite this equality as follows:

$$y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).$$

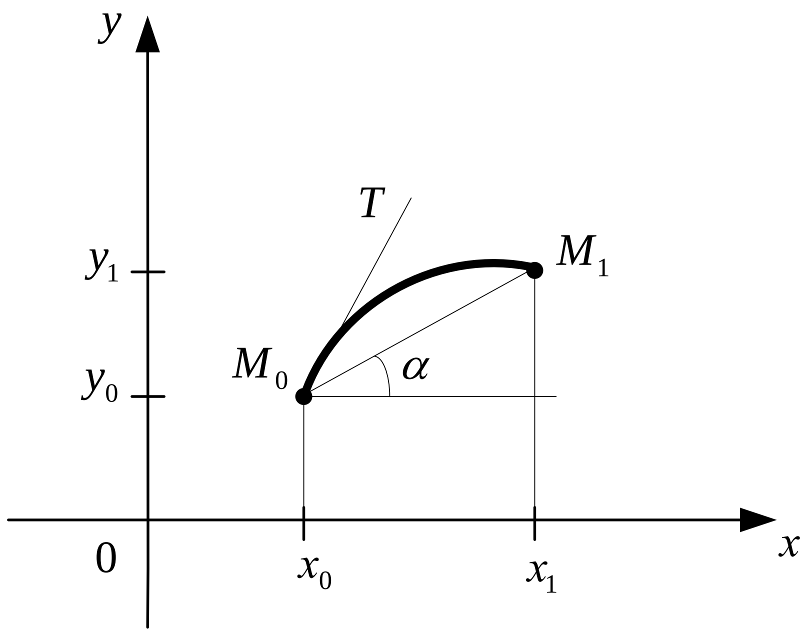

Fig. 1. **The graph of $f(x)$ and the secant passing through M_0 and M_1**

Now we give the following definition.

Definition 3. The straight line which is approached in the limit by the secant as $x_1 \to x_0$ will be called a tangent to the curve $y = f(x)$ at the point (in Fig. 1 that is denoted as M_0).

We can show that for the function $f(x)$ having the derivative at the point x_0 the tangent exists. Indeed, in this case $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0).$$

So the equation of the tangent to the graph of the function $y = f(x)$ at the point x_0 has the form:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Here

$$\tan\alpha = \lim_{x_1\to x_0} \tan\alpha_s = \lim_{x_1\to x_0} \frac{y_1 - y_0}{x_1 - x_0} = f'(x_0),$$

where α is the angle between the tangent and x-axis.

Thus we can give the following definition.

Definition 4. The value of the derivative $f'(x_0)$ is equal to the slope of the tangent line to the graph of the function $y = f(x)$ at the point with abscissa x_0.

Here it is convenient to present the definition of the normal to a curve.

Definition 5. The straight line passing through the point $M_0(x_0, y_0)$ perpendicularly to the tangent line at that point is called the normal to the curve at its point $M_0(x_0, y_0)$ (Fig. 2).

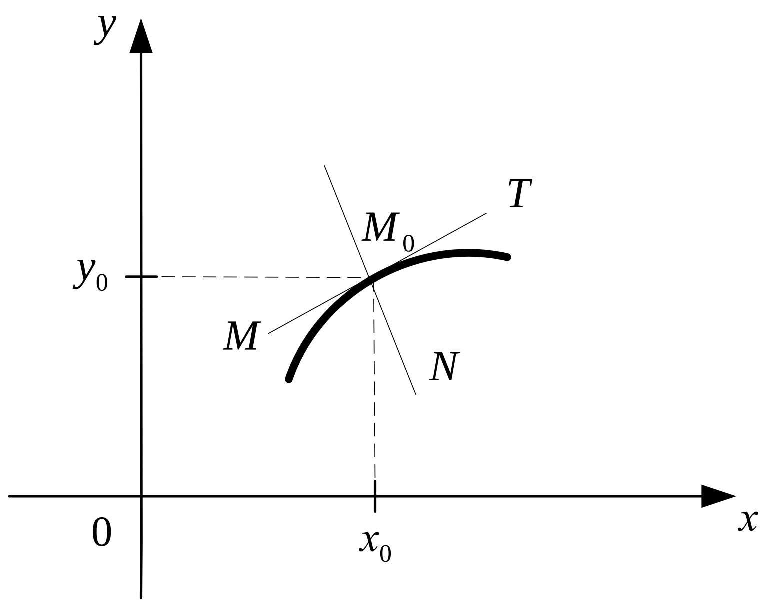

Fig. 2. **The normal M_0N to the curve of $f(x)$ and the tangent MT**

By the definition it follows that the slope of the normal to the curve at the point $M_0(x_0, y_0)$ being equal to $-\dfrac{1}{f'(x_0)}$, the equation of the normal can be written in the form:

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0).$$

(In Fig. 2 the normal is denoted by M_0N).

The notion of the differential of a function was explained above now let's introduce its geometrical interpretation (Fig. 3).

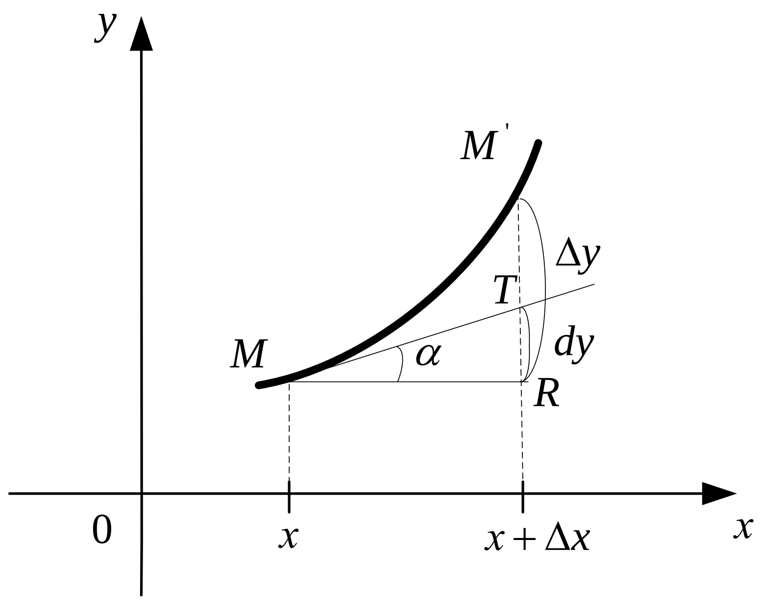

Fig. 3. **The geometrical interpretation of the differential of** $f(x)$

Since $f'(x) = \tan\alpha$ (α is the angle between the tangent MT and x-axis) the differential $dy = f'(x)dx$ is equal to the length of the line segment RT that is the differential dy of a function $y = f(x)$ at a point x is equal to the increment of the ordinate of the tangent line drawn to the graph of the function at its corresponding point.

The increment of the function $\Delta f(x)$ is equal to the increment of the of the ordinate of the graph of the function (i.e. to the line segment RM' in Fig. 3) and therefore the difference between the increment of the function and its differential is equal to the length of the line segment $M'T$ lying between the tangent and the graph. This line segment is an infinitesimal of higher order than the segment MR as $\Delta x \to 0$. For a concrete finite value of the increment Δx of the differential of a function may be greater or less than its increment.

4. Condition for Differentiability of a Function. Relation between the Notion of Differentiability of a Function and its Continuity

Definition 6. The function is called *a differentiable function* at the point x_0 if the derivative or the differential of the function exists at this point.

The process of finding the derivative or the differential of a function is referred to as the differentiation of the function.

Let's show the relation between the notion of differentiability of a function and its continuity, which is connected in the following theorem.

Theorem. A function $f(x)$ differentiable at a point x_0 is continuous at this point.

Let $\Delta f(x_0) = f'(x_0)\Delta x + o(\Delta x)$. Then

$$\lim_{x \to x_0} f(x) = f(x_0) + \lim_{\Delta x \to 0} \Delta f(x_0) =$$

$$= f(x_0) + f'(x_0) \lim_{\Delta x \to 0} \Delta x + \lim_{\Delta x \to 0} o(\Delta x) = f(x_0).$$

Since, $\lim\limits_{x \to x_0} f(x) = f(x_0)$ hence this differentiable function is continuous at the point.

The converse statement is not true. The continuity of the function does not involve the differentiability of a function.

For instance, the function $y = |x|$ is continuous throughout Ox but has no derivate at the point $x = 0$.

Indeed, we have $\Delta y = |x + \Delta x| - |x|$ and the substitution of $x = 0$ yields $\Delta y = |\Delta x|$ whence $\dfrac{\Delta y}{\Delta x} = \dfrac{|\Delta x|}{\Delta x}$.

We see that $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta x}{\Delta x} = 1$ for $\Delta x > 0$ and $\dfrac{\Delta y}{\Delta x} = -\dfrac{\Delta x}{\Delta x} = -1$ for $\Delta x < 0$. Therefore, the ratio $\dfrac{\Delta y}{\Delta x}$ has no limit as Δx tends to zero arbitrarily, which is equivalent to the nonexistence of the derivative at the point $x = 0$. This fact is also clear from the geometrical point of view: the graph of the function $y = |x|$ is a broken line (Fig. 4) with a corner point at the origin $(0; 0)$ and there is of course no tangent line to the graph of that point.

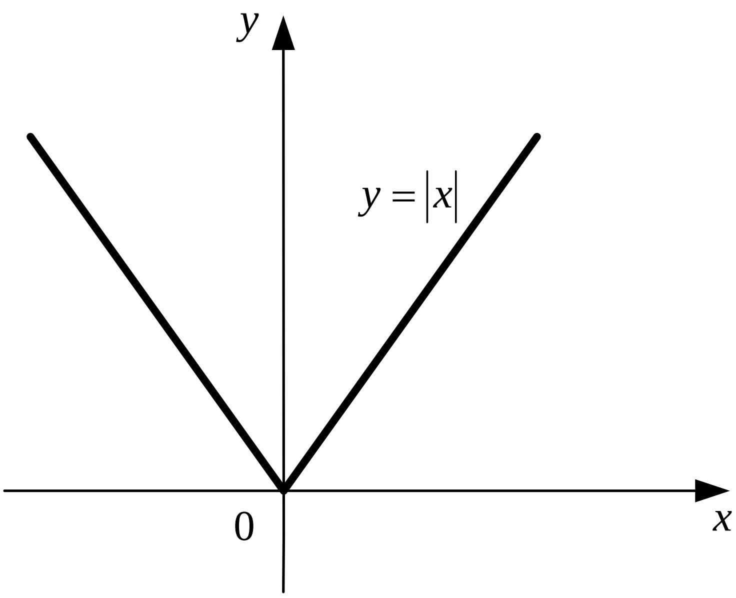

Fig. 4. **The graph of the function** $y = |x|$

A more general case of a corner point is presented in Fig. 5, where the curve has no single derivative but has two different derivatives at the point A – a left- and right-hand derivative

$$\lim_{\Delta x \to 0-0} \frac{\Delta y}{\Delta x} = k_1, \qquad \lim_{\Delta x \to 0+0} \frac{\Delta y}{\Delta x} = k_2.$$

The tangent rays emanate from this corner point with slope k_1 and k_2.

It can also happen that the graph of a continuous function has a tangent at a given point but the derivative of the function does not exist at that point.

This is the case when the tangent is perpendicular to the axis of abscissas.

In this case the function $f(x)$ is said to have at a given point x_0 an infinite derivative equal to $+\infty$ or $-\infty$ if

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = +\infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\infty .$$

Examples of such points are shown in Fig. 6.

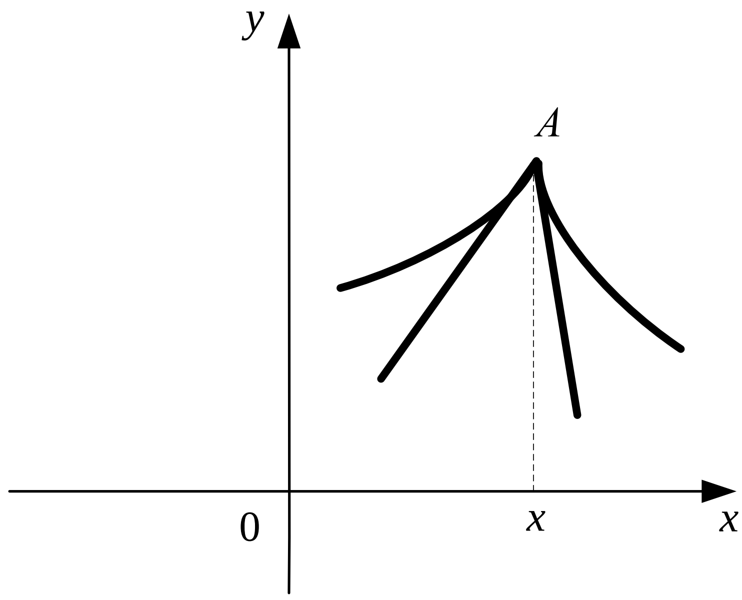

Fig. 5. **More general case of a corner point**

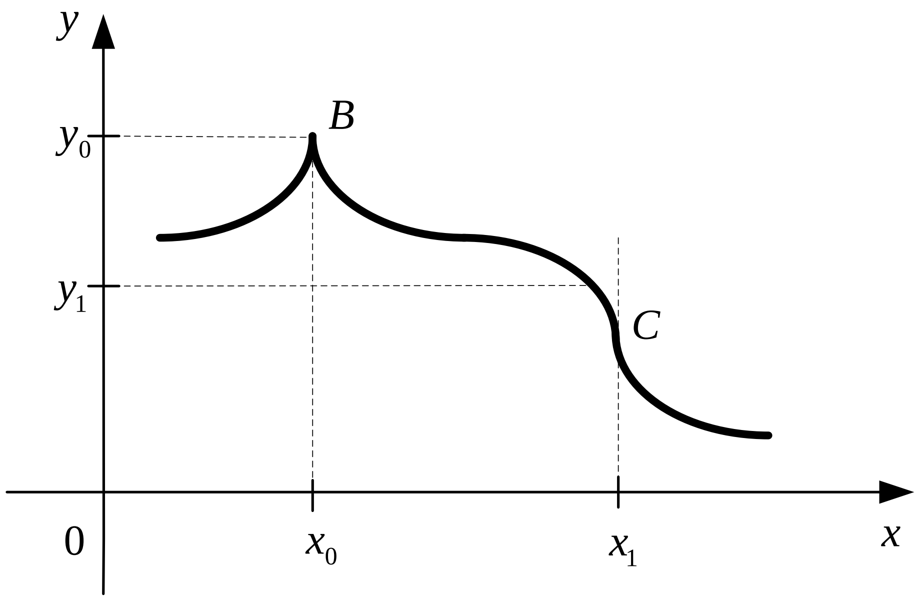

Fig. 6. **The function $f(x)$ with infinite derivatives**

The point B is called a cuspidal point or cusp. The cusp is a particular case of a corner point. And C is called a point of inflection. Both these points have vertical tangent lines.

At last it should be remarked that the function is termed a differentiable on an interval if it is differentiable at every point of this interval.

5. Properties of Derivatives and Differentials of Functions of One Variable

1. If the function $f(x)$ is equal to a constant then its derivative is equal to zero, that is $f(x)=C=const$, $f'(x)=C'=0$. It follows from the fact that function $f(x)=C$ has no increment for all variations of x,

$$\Delta f(x)=\Delta C=0.$$

The rest properties are included in the following theorems.

Theorem 1. Let functions $u(x)$ and $v(x)$ have derivatives at a point x then at this point also exist the derivatives of function $C_1u(x)+C_2v(x)$ (C_1 and C_2 constants), $u(x)\cdot v(x)$, $\frac{u(x)}{v(x)}$ (for $v(x)\neq 0$).

Theorem 2. $[C_1u(x)+C_2v(x)]'=C_1u'(x)+C_2v'(x)$.

The derivative of a sum of two or a finite number of functions is equal to the sum of the derivatives of the summands.

Theorem 3. $[u(x)\cdot v(x)]'=u'(x)\cdot v(x)+u(x)\cdot v'(x)$.

The derivative of the product of two functions is equal to the sum of the product of the derivative of the first function by the second function and the product of the derivative of the second function by the first function.

Theorem 4. $\left[\frac{u(x)}{v(x)}\right]'=\frac{u'(x)\cdot v(x)-u(x)\cdot v'(x)}{(v(x))^2}$, $(v(x)\neq 0)$.

The derivative of the quotient of two functions is equal to the function whose denominator is equal to the square of the divisor and the numerator is equal to the difference between the product of the derivative of the dividend by the divisor and the product of the dividend by the derivative of the divisor.

This theorem is given without proof but it should be noted this proof is based on the definition of the derivative. There are corresponding formulas for differentials.

Since $(u+v)' = u' + v'$, then multiplying both sides of the relation by dx we obtain

1. $d(u+v) = du + dv$.

Since $(u \cdot v)' = u' \cdot v + u \cdot v'$, then multiplying of both sides by dx leads to

2. $d(u \cdot v) = vdu + udv$ and in particular $d(Cu) = Cdu$, $dC = 0$.

Taking the formula $\left[\dfrac{u(x)}{v(x)}\right]' = \dfrac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(v(x))^2}$ and multiplying its both members by dx we receive $d\left(\dfrac{u}{v}\right) = \left[\dfrac{u}{v}\right]' dx = \dfrac{u'dx \cdot v - u \cdot v'dx}{v^2} = \dfrac{vdu - udv}{v^2}$.

All these properties for derivatives and differentials are recommended for remembering.

6. Differentiating Composite Functions. Invariance of the form of the Differential

The notion of a composite function was given in the guidelines "Introduction to Analysis". There is the following theorem for differentiation of such a function.

Theorem. The derivative of a function is equal to the product of the derivative of the given function with respect to the intermediate argument by the derivative of this argument with respect to the independent variable.

Let $y = f(u)$ and $u = \varphi(x)$. It is required to prove that $y' = f'(u) \cdot u' = f'(u) \cdot \varphi'(x)$.

Let x receive an increment Δx. This results in an increment Δu of the intermediate argument $u = \varphi(x)$ which in its turn, generates an increment Δy

of the magnitude y. To find y' we must compute $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$ as $\Delta x\to 0$. Let us represent the ratio $\dfrac{\Delta y}{\Delta x}$ in the form $\dfrac{\Delta y}{\Delta x}=\dfrac{\Delta y}{\Delta u}\cdot\dfrac{\Delta u}{\Delta x}$. According to the rules for passing to the limit in a product we can write

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}\right)=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x},$$

where $\Delta u\to 0$, as $\Delta x\to 0$, since $u=\varphi(x)$ is a continuous function. Since

$$\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}=f'(u)\quad\text{and}\quad\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\varphi'(x)$$

we arrive at the desired formula.

Applying this theorem for the differential of a composite function, we have

$$df(\varphi(x))=[f(\varphi(x))]'dx=f'(u)\cdot\varphi'(x)dx=f'(u)du,$$

where $u=\varphi(x)$. The obtained relationship shows that the differential and the derivative of the composite function $y=f(\varphi(x))$ are related to the differential of the dependent variable $u=\varphi(x)$ in the same way as is the case when u is an independent variable. This property is known as invariance of the form of the notation of a differential.

7. Differentiating Inverse Function and Functions Represented Parametrically

Let $y=f(x)$ and $x=\varphi(y)$ be a pair of mutually inverse functions. The function $x=\varphi(y)$ can be obtained by resolving the equation $y=f(x)$ with respect to x. For definiteness let the derivative $f'(x)$ be known and it does not turn into zero. In this case the function $x=\varphi(y)$ is also the continuous function.

In order to find the derivative $x'_y=\varphi'(y)$ we use the representation of the derivatives in term of differentials. Then we obtain

$$x'_y=\frac{dx}{dy}=\frac{1}{dy/dx}=\frac{1}{f'(x)}.$$

Similarly, if $\varphi'(y)\neq 0$ then $f'(x)=\dfrac{1}{\varphi'(y)}$. Briefly, the derivatives of two inverse function are the reciprocates of each other that is

$$y'_x=\frac{1}{x'_y} \quad \text{or} \quad x'_y=\frac{1}{y'_x}.$$

Now let us consider parametric representation of functions. Let function

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}, \tag{*}$$

be functions of one and the same variable t.Such representation is called parametric representation of functions. The specification of these functions yields a functional relationship between the variables x and y. For with each value of t (belonging to the given domain) this system (*) associates same value of x and y and thus generates a correspondence between x and y, t is called a parameter.

Let functions $\varphi(t)$ and $\psi(t)$ be differentiable on an interval $[t_1, t_2]$ and $\varphi'(t)\neq 0$. Then the property for the differential of a composite function, which had been regarded above, may be conveniently used for computing the derivative of a function represented parametrically. Since $y'_x=\dfrac{dy}{dx}$ for $x=\varphi(t)$ and $y=\psi(t)$ we have $dx=\varphi'(t)$ and $dy=\psi'(t)$ then

$$y'_x = \frac{\psi'_t}{\varphi'_t} = \frac{y'_t}{x'_t}.$$

8. Derivatives of Elementary Functions

We have considered some rules for computing derivatives of functions of one variable. These rules enable us to compute derivatives of any elementary functions from a knowledge of derivatives of simple elementary functions. Let us prove that for a derivative of simple elementary functions the following formulas are valid.

The derivatives of the power function and basic trigonometric functions:

1. $\left(x^n\right)' = nx^{n-1}$,

$$\left(x^n\right)' = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{nx^{n-1} + o(\Delta x)}{\Delta x} = nx^{n-1}.$$

2. $(\sin x)' = \cos x$,

$$(\sin x)' = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sin\left(\Delta x/2\right)\cos\left(x + \Delta x/2\right)}{\Delta x} = \cos x,$$

as $\lim\limits_{\Delta x \to 0} \cos\left(x + \Delta x/2\right) = \cos x$, $\lim\limits_{\Delta x \to 0} \dfrac{\sin\left(\Delta x/2\right)}{\Delta x/2} = 1$ (the first remarkable limit).

3. $(\cos x)' = -\sin x$,

$$(\cos x)' = \left(\sin\left(\pi/2 - x\right)\right)' = -\cos\left(\pi/2 - x\right) = -\sin x.$$

4. $(\tan x)' = 1/\cos^2 x$,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos x} = 1/\cos^2 x.$$

5. $(\cot x)' = -\,1\big/\sin^2 x\,.$

This derivative is obtained similarly as the derivative of $\tan x$.

The derivatives of inverse trigonometric functions can be found by means of the expressions for the derivatives of trigonometric functions, using the rule for differentiating inverse functions.

6. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}\,.$

If $y = \arcsin x$ then $x = \sin y$ and $x'_y = \cos y$, therefore

$$y'_x = (\arcsin x)' = \frac{1}{x'_y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}\,.$$

7. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}\,.$

8. $(\arctan x)' = \dfrac{1}{1+x^2}\,.$

9. $(arc\cot anx)' = -\dfrac{1}{1+x^2}\,.$

Now let us consider the derivatives of logarithmic and exponential functions.

10. $(\ln x)' = \dfrac{1}{x},$

$$(\ln x)' = \lim_{\Delta x\to 0}\frac{\ln(x+\Delta x)-\ln x}{\Delta x} = \lim_{\Delta x\to 0}\frac{\ln\left(1+\Delta x\big/x\right)}{\Delta x} =$$

$$= \frac{1}{x}\lim_{\Delta x\to 0}\frac{\ln\left(1+\Delta x\big/x\right)}{\Delta x\big/x} = \frac{1}{x}\lim_{\Delta x\to 0}\left(\ln\left(1+\Delta x\big/x\right)\right)^{\Delta x/x} = \frac{1}{x}\ln e = \frac{1}{x}.$$

According to the property that symbols of the logarithm and the limit can be interchanged if the expression under the sign of logarithm possesses a limit as $\Delta x \to 0$ and recalled that

$$\lim_{\alpha(x)\to 0}\left(\ln\left(1+\alpha(x)\right)\right)^{1/\alpha(x)} = e,$$

we obtain

$$= \frac{1}{x}\ln \lim_{\Delta x\to 0}\left(1+\frac{\Delta x}{x}\right)^{x/\Delta x} = \frac{1}{x}\ln e = \frac{1}{x}.$$

11. $(\log_a x)' = \dfrac{1}{x\ln a}$, $\log_a x = \dfrac{\ln x}{\ln a}$, then $(\log_a x)' = \left(\dfrac{\ln x}{\ln a}\right)' = \dfrac{1}{x\ln a}$.

12. $(a^x)' = a^x \ln a$, $y = a^x$, then $\ln y = x\ln a$ whence $x = \dfrac{\ln y}{\ln a}$ and

$$y'_x = (a^x)' = \frac{1}{x'_y} = \frac{\ln a}{(\ln y)'} = y\ln a = a^x \ln a.$$

If $a = 1$ then $(e^x)' = e^x$.

The underlined formulas for derivatives of the simple elementary functions are insistently recommended for remembering

Now let us consider several examples for computing derivatives using the properties of the derivatives, known rules for differentiating and the formulas for differentiating of simple elementary functions.

Example 1. Find the derivative: $y = \sqrt{x}\ln x + \dfrac{\sin x}{e^x}$.

Solution. Applying the properties of derivatives, in particular theorem of Sec. 5, items 2,3 and 4 and derivatives of simple elementary function, we get

$$y' = \left(\sqrt{x}\ln x\right)' + \left(\frac{\sin x}{e^x}\right)' = \left(\sqrt{x}\right)'\ln x + \sqrt{x}\left(\ln x\right)' + \frac{\left(\sin x\right)' e^x - \left(e^x\right)'\left(\sin x\right)}{\left(e^x\right)^2} =$$

$$= \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x} + \frac{\cos x e^x - e^x \sin x}{e^{2x}} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + \frac{\cos x - \sin x}{e^x}.$$

Finally

$$y' = 0{,}5x^{-0{,}5}\left(\ln x + 2\right) + e^{-x}\left(\cos x - \sin x\right).$$

Example 2. Find the derivative: $y = \ln \tan x$.

Solution. It is a composite function. Therefore using the rule of differentiating of the composite function which was given in Sec. 6 we consider $\tan x$ as the argument of the logarithmic function. Then we find the derivative of logarithmic and multiply it by the derivative of the argument, that is

$$y' = \left(\ln \tan x\right)' = \frac{\left(\tan x\right)'}{\tan x} = \frac{1}{\tan x \cos^2 x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x}.$$

Example 3. Find the derivative: $y = \ln \sin\left(x^2 + 1\right)$.

Solution. Acting analogously to example 2 we have

$$y' = \left(\ln \sin\left(x^2+1\right)\right)' = \frac{\left(\sin\left(x^2+1\right)\right)'}{\sin\left(x^2+1\right)} = \frac{2x\cos\left(x^2+1\right)}{\sin\left(x^2+1\right)} = 2x\cot\left(x^2+1\right).$$

Example 4. Find the derivative: $y = \arctan\sqrt{x}$.

Solution. Let us find this derivative using the rule for differentiating of the inverse function (sec. 7) . Then $x = \tan^2 y$ and

$$x'_y = \left(\tan^2 y\right)' = 2\tan y\left(\tan y\right)' = \frac{2\tan y}{\cos^2 y},$$

Whence

$$y'_x = \frac{1}{x'_y} = \frac{\cos^2 y}{2\tan y}.$$

Since $\sqrt{x} = \tan y$ and $\cos^2 y = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+\left(\sqrt{x}\right)^2} = \dfrac{1}{1+x}$, then finally we have $y'_x = \dfrac{1}{2\sqrt{x}(1+x)}$.

Example 5. Find the derivative: $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ $(0 \le t \le 2\pi)$.

Solution. It is parametric representation of the ellipse because

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1.$$

Using formula (**) of Sec. 7 for the function represented parametrically, we obtain

$$y'_x = \frac{y'_t}{x'_t} = \frac{(b\sin t)'}{(a\cos t)'} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot ant.$$

9. Derivatives of Implicit Functions

Suppose that y is an implicit function of x which means that it is specified by an equation connecting the independent variable x and the function y, the equation can not be solved with respect to y, i.e. $F(x, y) = 0$.

Then the derivative of this function (provided it exists) can usually be found by differentiating (with respect to x) both sides of the equation. In this differentiation it is necessary to take into account that y is a function of x that is $y = y(x)$ (specified by this equation). We will illustrate by examples the practical significance of this rule.

Example 1. Let us determine the derivative of the function y specified by the equation $y^2 - 25x = 0$ (this is the equation of the parabola).

Solution. Differentiating with respect to x and taking into account that y is a function of x we receive $2yy' - 25 = 0$, whence $y' = \dfrac{25}{2y}$.

In this example it is not difficult to find the explicit expression for y, namely $y = 5\sqrt{x}$ or $y = -5\sqrt{x}$.

The differentiation of these functions yields $y' = \dfrac{5}{2\sqrt{x}}$ and $y' = -\dfrac{5}{2\sqrt{x}}$ that is coincided with the result just obtained.

Consider another example.

Example 2. Find the derivative: $ye^{2y} + \ln(x - y) = 0$.

Solution. Differentiating with respect to x we get

$$y'e^{2y} + ye^{2y}2y' + \frac{1 - y'}{x - y} = 0 \text{ or } y'\left(e^{2y} + 2ye^{2y} - \frac{1}{x - y}\right) = -\frac{1}{x - y},$$

$$y' = \left(e^{2y} + 2ye^{2y} - \frac{1}{x - y}\right) = -\frac{1}{\left(e^{2y}(x - y)(1 + 2y) - 1\right)}.$$

Thus, the derivative of any implicit function specified by an equation involving elementary functions can be determined according to the known differentiation rules irrespective of whether it is possible to respect the function explicitly. In the general case the derivative of such a function is expressed in terms of the independent variable and the function itself.

10. Logarithmic Differentiation

When computing the derivative of a function which can be represented in the form

$$y = f(x) = \varphi(x)^{\psi(x)} \quad (\varphi(x) > 0),$$

where the base and the exponent are both functions of the independent variable x is referred to as a composite exponential (or power exponential) function, it is convenient to use the so-called logarithmic derivative, that is the derivative of the natural logarithm. Indeed, for

$$\ln y = \ln f(x) = \ln \varphi(x)^{\psi(x)} = \psi(x) \ln \varphi(x)$$

as a result of this operation, is

$$(\ln y)' = (\ln f(x))' = \frac{f'(x)}{f(x)} = \frac{y'}{y} = \psi'(x) \ln \varphi(x) + \psi(x) \frac{\varphi'(x)}{\varphi(x)},$$

then

$$y' = y\left(\psi'(x) \ln \varphi(x) + \psi(x) \frac{\varphi'(x)}{\varphi(x)}\right) = \varphi(x)^{\psi(x)}\left(\psi'(x) \ln \varphi(x) + \psi(x) \frac{\varphi'(x)}{\varphi(x)}\right).$$

Logarithmic differentiation is not only applicable to finding the derivative of a composite exponential function but also to some other problems, for instance, to computing derivatives of a product of powers of some elementary functions, a quotient of algebraic and transcendental functions and others.

Example 1. Find the derivative: $y = \sin x^{\cos x} \quad (0 < x < \pi)$.

Solution. First we find a logarithm of this function and then we use the logarithmic derivative $\ln y = \ln \sin x^{\cos x} = \cos x \cdot \ln \sin x$ and

$$(\ln y)' = \left(\ln \sin x^{\cos x}\right)' = (\cos x \cdot \ln \sin x)' \text{ or } \frac{y'}{y} = -\sin x \cdot \ln \sin x + \frac{\cos^2 x}{\sin x}$$

whence $y' = \sin x^{\cos x}\left(\dfrac{\cos^2 x}{\sin x} - \sin x \cdot \ln \sin x\right)$.

11. Applying Differential to Approximate Calculations

The application of the differential to approximate calculations is based on the replacement of the increment $\Delta y = f(x_0 + \Delta x) - f(x_0)$ of a given function $f(x)$, which may depend on $\Delta x = dx$ in a complicate manner, by the simpler expression $f'(x_0)dx$ (the differential) found by differentiation.

Thus for small values of dx we write

$$\Delta y = f(x_0 + \Delta x) - f(x_0) = dy + o(\Delta x) = f'(x_0)dx + o(\Delta x),$$

with a small relative error we can put: $\Delta y \approx dy$, then

$$f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)dx.$$

This approximate equality can be immediately used to solving the following value $f(x_0 + \Delta x)$. So $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)dx$, ($\Delta x = dx$).

Example 1. Compute $\sqrt[3]{0.988}$.

Solution. For approximate calculation of this expression we consider the function $y = f(x) = \sqrt[3]{x}$ and suppose that $f(x_0 + \Delta x) = \sqrt[3]{0.988}$.

The initial point $x_0 = 1$ and $f(x_0) = f(1) = 1$, then $dx = -0.012$.

The differential of this function $dy = f'(x_0)dx = \frac{1}{3}x_0^{-\frac{2}{3}}dx$. Substituting $x_0 = 1$ we obtain $dy = \frac{1}{3} \cdot 1 \cdot (-0.012)$. Considering that $\Delta y \approx dy$ we get $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)dx$, whence

$$\sqrt[3]{0.988} \approx \sqrt[3]{1} - \frac{1}{3} \cdot 0.012 = 1 - 0.004 = 0.996.$$

Example 2. Compute $\ln 1.1$.

Solution. In this case we suppose $f(x) = \ln x$ and $f(x_0 + \Delta x) = \ln 1.1$.

Considering that $x_0 = 1$ and $dx = 0.1$, then $f'(x) = \frac{1}{x}$ and $f'(x_0) = 1$. Since $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)dx$, we obtain

$$\ln 1.1 = \ln 1 + 1 \cdot 0.1 = 0 + 0.1 = 0.1.$$

12. Derivatives and Differentials of Higher Orders of Functions of One Variable

The derivative of a function of one variable $y = f(x)$ is some function of the variable x. Therefore, we can try to find its derivative. The obtained function if it exists is called the second derivative, or the derivative of the second order of the function $f(x)$. Analogously, by induction, we can introduce the derivatives of higher orders.

Since $y' = f'(x)$ is the derivative of the first order, then

$$y'' = (y')' = f''(x) = \frac{d^2 f(x)}{dx^2}.$$

In a similar way we obtain the derivative of the *n*-th order:

$$y^{(n)} = \left(y^{(n-1)}\right)' = f^{(n)}(x) = \frac{d^{(n)} f(x)}{dx^n}.$$

Physical Interpretation of the Second Derivative. Let $S = F(t)$ denote the part covered by a material point during time t. Then it had been said above (Sec. 1), then $\lim\limits_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} = F'(t)$ is the instantaneous velocity of the point at time instant t. In the same way, we can make sure that the derivative $\lim\limits_{\Delta t \to 0} \frac{F'(t + \Delta t) - F'(t)}{\Delta t} = F''(t) = \frac{d^2 F(t)}{dt^2}$ is the instantaneous acceleration of the point at time instant t.

Let us now introduce the notion of differentials of higher orders. With dx fixed, the differential $dy = df(x) = f'(x)dx$ is a function of the variable x. The differential of this function is called the differential of the second order and is denoted as follows:

$$d^2 y = d^2 f(x).$$

If $y = f(x)$, then

$$dy = f'(x)dx,\ d^2 y = d(dy) = d(f'(x)dx) = d(f'(x))dx = f''(x)dx^2.$$

Analogously, using the principle of induction, we can also define differential of higher orders:

$$d^n y = d\left(d^{(n-1)} y\right) = f^{(n)}(x)dx^n.$$

Therefore, instead of $f^{(n)}(x)$ we may use the notation $\dfrac{d^n y}{dx^n}$.

Differentials of higher orders do not possess the invariance of the form of the notation which is possessed by the differential of the first order. For instance, $d^2 f(\varphi(x)) \neq f''(\varphi(x)) d^2 \varphi(x)$.

Let us consider examples on finding derivatives and differentials of higher orders.

Example 1. Let us derive general formulas for derivatives of order n for some elementary functions.

We find by induction:

$$\left(x^\alpha\right)^{(n)} = \alpha(\alpha-1)(\alpha-2)(\alpha-3)\ldots(\alpha-n+1)x^{(\alpha-n)},$$

$$\left(a^x\right)^{(n)} = a^x(\ln a)^n, \quad \left(e^x\right)^{(n)} = e^x,$$

$$(\sin x)^{(n)} = \sin\left(x + \frac{\pi n}{2}\right), \quad (\cos x)^{(n)} = \cos\left(x + \frac{\pi n}{2}\right),$$

$$y = \ln(1+x), \quad y' = \frac{1}{1+x}, \quad y'' = -\frac{1}{(1+x)^2}, \ \ldots,$$

$$y^{(n)} = (\ln(1+x))^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Example 2. Find the differential of the second order for the function $y = \frac{1}{x} - \frac{1}{x^2}$.

Solution. Using the general formula $d^2y = y'' dx^2$ we find $y'' = -\frac{1}{x^2} + \frac{2}{x^3}$, then

$$d^2y = \left(-\frac{1}{x^2} + \frac{2}{x^3}\right)dx^2 = \frac{2-x}{x^3}dx^2.$$

13. Local Extremum. Mean-Value Theorems

We begin with the following definition.

Definition 1. We will say that the function $y = f(x)$ reaches a local maximum (minimum) at the point $x = x_0$ if there is a neighborhood of this point $U_{x_0} = (x_0 - \varepsilon; x_0 + \varepsilon)$ where the inequality is fulfilled.

$f(x_0) \geq f(x), \ \forall x \in U_{x_0}$ (for all x belonging to U_{x_0}) (respectively $f(x_0) \leq f(x), \ \forall x \in U_{x_0}$).

A local extremum is a general term for a local maximum or a local minimum.

And now we introduce mean-value theorems, namely Fermat′s, Rolle′s, Cauchy′s and Lagrange′s theorems.

Fermat′s theorem. If a function $f(x)$ reaches a local extremum at a point x_0 and the derivative $f'(x_0)$ of the function $f(x)$ at the point x_0 exists then it is necessarily equal to zero: $f'(x_0) = 0$.

Proof. For definiteness we suppose that $f(x)$ has at the point x_0 a local maximum. By the definition of the derivative we have $f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

Since $f(x_0)\ge f(x)$, $\forall x\in U_{x_0}$, then for sufficiently small $\Delta x>0$ $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\le 0$, whence passing to the limit as $\Delta x\to 0$, we receive that

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\le 0. \qquad (1)$$

If $\Delta x<0$ then $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\ge 0$, therefore the limit of this in-equality as $\Delta x\to 0$ equals to

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\ge 0. \qquad (2)$$

From the relationship (1) and (2) it follows that $f'(x_0)=0$.

This theorem is illustrated in Fig 7.

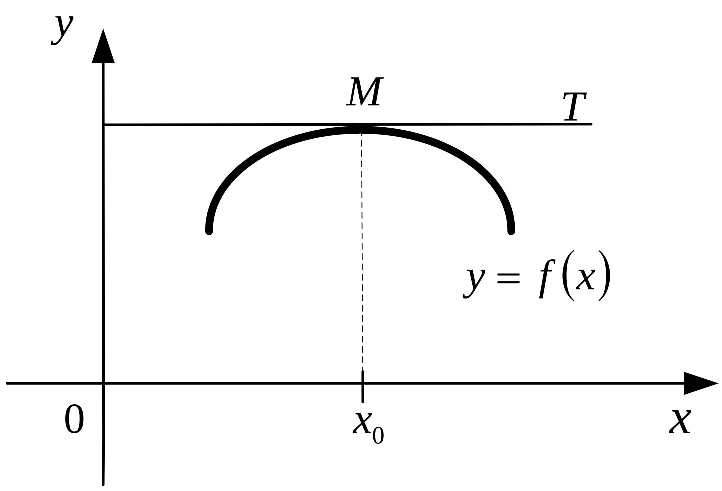

Fig. 7. **The illustration for Fermat's theorem**

Here MT is the tangent at the point M with abscissa x_0. The slope of this tangent is equal to zero, that is $f'(x_0)=0$.

We give the rest theorem without proof but we introduce the geometrical interpretation of Rolle's and Lagrange's theorem.

Rolle's theorem. If the function $y=f(x)$ is continuous on a closed interval $[a;b]$, differentiable at all the interior points of an open interval $(a;b)$ and assumes equal values at the end points of this interval $f(a)=f(b)$, then there is at least one point c inside the interval $(a;b)$ at which $f'(c)=0$.

This theorem has a simple geometrical significance. If the conditions of this theorem are fulfilled, then on the graph of the function $y=f(x)$ there is the point $(c;f(c))$ at which the tangent line is parallel to x-axis (Fig. 8).

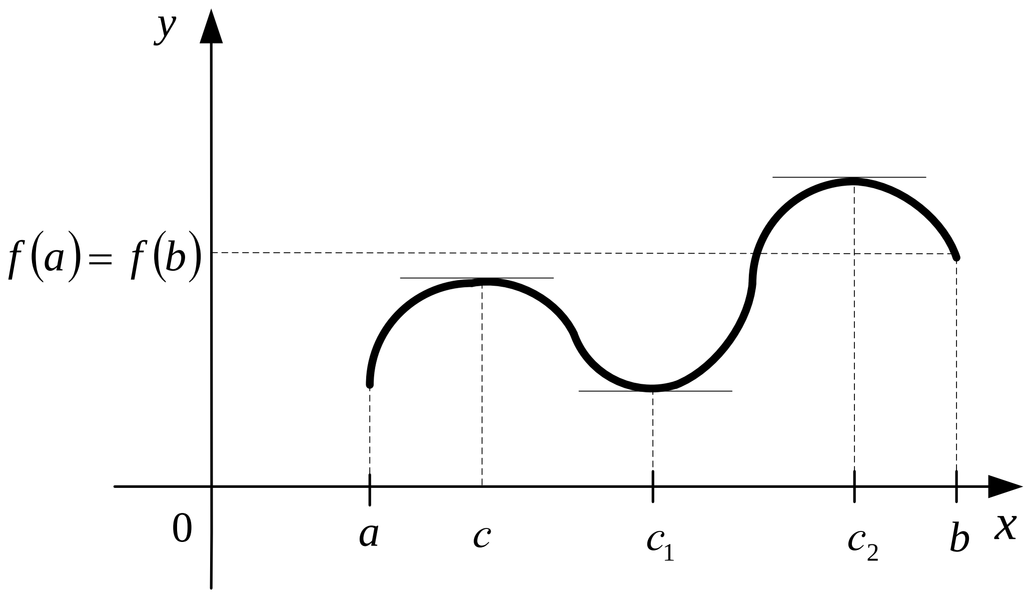

Fig. 8. **The function $y=f(x)$ and the tangent line is parallel to x-axis**

Cauchy's theorem. Let functions $f(x)$ and $g(x)$ be continuous on a closed interval $[a;b]$ and haves derivatives not vanishing simultaneously on an open interval $(a;b)$ and $f(a)\neq f(b)$.

Then on the interval $(a;b)$ there is a point c for which the following equality holds:

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)} \quad (a<c<b).$$

The corollary of this theorem if $g(x)=x$ is Lagrange's theorem for finite increments.

Lagrange′s theorem. Let the function $f(x)$ be continuous on a closed interval $[a;b]$ and have the derivative on an open interval $(a;b)$. Then on the interval $(a;b)$ there is a point c for which the following equality holds:

$$f(b)-f(a)=f'(c)(b-a) \quad (a<c<b).$$

This theorem has the following geometrical interpretation, if we rewrite it in the form

$$\frac{f(b)-f(a)}{b-a}=f'(c) \ (a<c<b).$$

The left-hand side of this equality is the slope to x-axis of the chord MN joining the points $M(a;f(a))$ and $N(b;f(b))$ of the graph of the function $y=f(x)$. The right-hand side of the equality is the slope of the tangent to the graph at a some interior point with the abscissa $c\in(a;b)$. By the Lagrange's theorem it follows that there is the point c on the curve that the tangent line to the graph of the function at this point is parallel to the chord MN (fig. 9).

14. Evaluation of Indeterminate Forms. L′Hospital′s Rule ′

We will say that the ratio $\frac{f(x)}{g(x)}$ presents the indeterminate form $\left\{\frac{0}{0}\right\}$ as $x\to a$ if $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$. To evaluate this indeterminate form it means to find $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ if it exists.

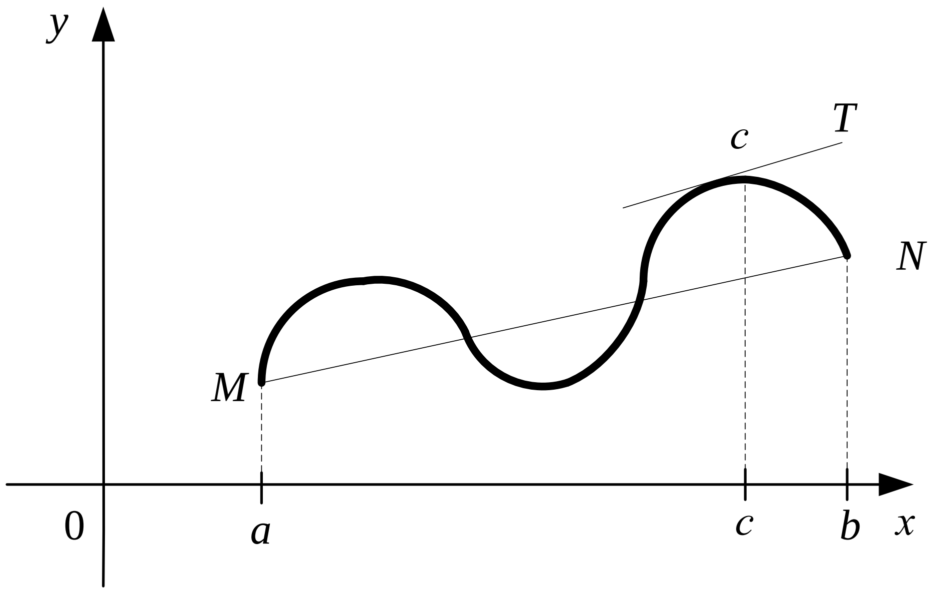

Fig. 9. **The graph of the function at the point is parallel to the chord** MN

There is the so-called L′Hospital′s rule for evaluating these indeterminate forms which is concluded in the following theorem.

Theorem. Let functions $f(x)$ and $g(x)$ be continuous and have first derivatives in the neighborhood of a point a except possibly the point a itself. If the functions $g(x)$ and $g'(x)$ are not equal to zero in the indicated neighborhood, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ and the limit $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, then an equal limit $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ also exists.

For indeterminate forms of the type $\left\{\dfrac{\infty}{\infty}\right\}$ there is the second L′Hospital′s rule which states that if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ when the latter exists.

We have presented this theorem without proof but we note that they can be proved with the aid of the mean-value theorem.

Remark 1. If the expression $\dfrac{f'(x)}{g'(x)}$ is the indeterminate form of the type

$\left\{\frac{0}{0}\right\}$ or $\left\{\frac{\infty}{\infty}\right\}$ and functions $f(x)$ and $g(x)$ satisfy to the conditions of the L'Hospital's rules, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}.$$

It should be stressed that if the third limit exists, then the second and the first limits also exist.

Remark 2. The indeterminate forms $\{0\cdot\infty\}$, $\{\infty-\infty\}$, $\{1^{\infty}\}$ and $\{0^{0}\}$ are reduced to the indeterminate forms $\left\{\frac{0}{0}\right\}$ and $\left\{\frac{\infty}{\infty}\right\}$ by means of algebraic transformations.

Let us consider examples.

Example 1. Evaluate $\lim\limits_{x\to 0}\frac{\sin 3x}{\sin 4x}$.

Solution. Here we have the indeterminate form $\left\{\frac{0}{0}\right\}$. Using L′Hospital′s rule we obtain

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}=\left\{\frac{0}{0}\right\}=\lim_{x\to 0}\frac{(\sin 3x)'}{(\sin 4x)'}=\lim_{x\to 0}\frac{3\cos 3x}{4\cos 4x}=\frac{3}{4}.$$

Example 2. Find $\lim\limits_{x\to a}\frac{x^2-a^2}{\sin(x-a)}$.

Solution. Here we have the indeterminate form $\left\{\frac{0}{0}\right\}$. Using L'Hospital's rule we obtain

$$\lim_{x\to a}\frac{x^2-a^2}{\sin(x-a)}=\left\{\frac{0}{0}\right\}=\lim_{x\to a}\frac{\left(x^2-a^2\right)'}{(\sin(x-a))'}=\lim_{x\to a}\frac{2x}{\cos(x-a)}=\frac{2a}{1}=2a.$$

This result is obtained analogously to the previous example.

Example 3. Find $\lim\limits_{x\to\frac{\pi}{2}}(1+\cos x)^{2/\cos x}$.

Solution. In this case we have the indeterminate form $\{1^{\infty}\}$. To evaluate it we can use the substitution $y=\frac{\pi}{2}-x$ then $y\to 0$ as $x\to\frac{\pi}{2}$. And we rewrite this limit in terms of a new variable

$$\lim_{x\to\frac{\pi}{2}}(1+\cos x)^{2/\cos x}=\lim_{y\to 0}(1+\sin y)^{2/\sin y}=e^{2},$$

since $\lim\limits_{\alpha(x)\to 0}(1+\alpha(x))^{k/\alpha(x)}=e^{k}$.

15. Increase and Decrease of a Function. Necessary and Sufficient Conditions for an Extrimum of the Function

Let us apply the notion of the derivative to studying the behavior of a function. Consider the value of a function of one variable $y=f(x)$ in some interval $[a;b]$ and in some ε-neighborhood of a fixed point x_0. Let us denote, as before

$$\Delta y=f(x_0+\Delta x)-f(x_0).$$

Definition 1. A function is called increasing (decreasing) in an interval $[a;b]$ if for any x_1 and x_2 within this interval $f(x_1)<f(x_2)$ ($f(x_1)>f(x_2)$) when $x_1<x_2$ (Fig. 10, 11).

Definition 2. We will say that the function $y=f(x)$ increases (decreases) at a point x_0 if there is $\varepsilon>0$ such that $\frac{\Delta y}{\Delta x}>0$ $\left(\frac{\Delta y}{\Delta x}<0\right)$ for $0<|x|<\varepsilon$, that is, the quantities Δy and Δx have like (unlike) signs (Fig. 12, 13).

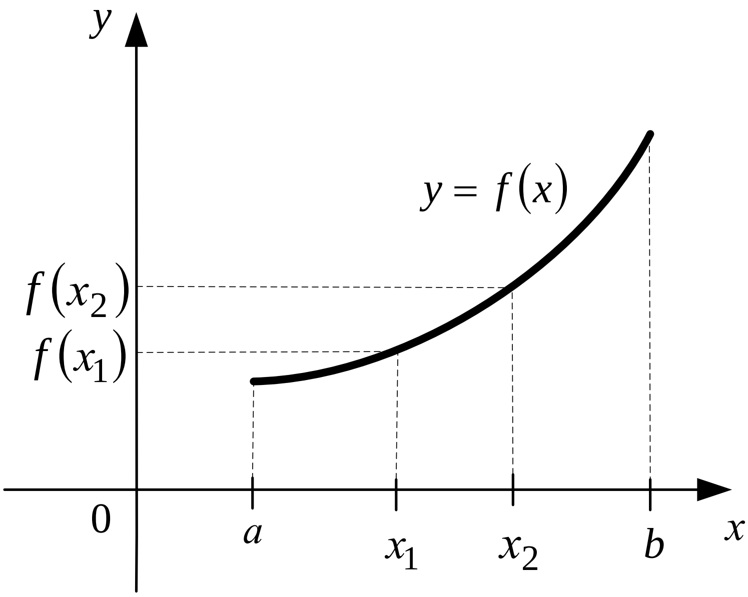

Fig. 10. **A function is increasing in an interval** $[a;b]$

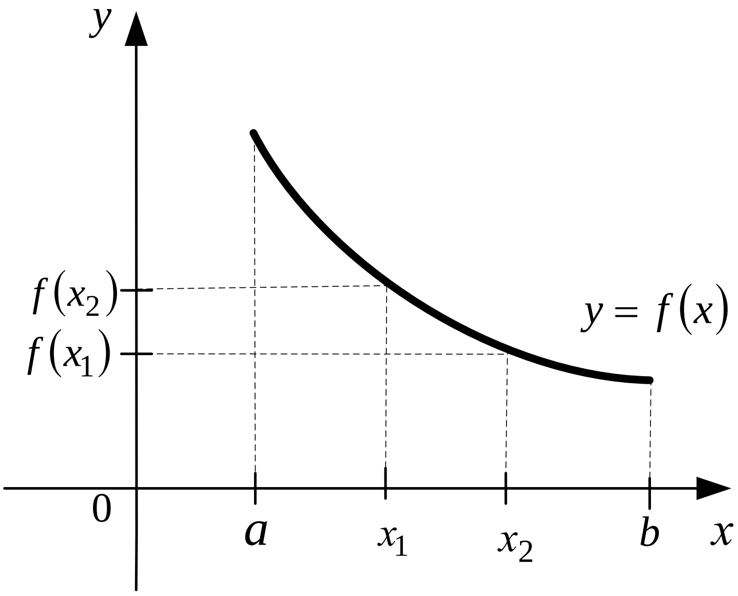

Fig. 11. **A function is decreasing in an interval** $[a;b]$

Basing on the latest definition we can give another definition for local extremum of a function.

Definition 3. The function $y = f(x)$ reaches a local maximum (minimum) of the point x_0 if there is $\varepsilon > 0$ such that $\Delta y \leq 0$ $(\Delta y \geq 0)$ for $|x| < 0$ (Fig. 12, 13).

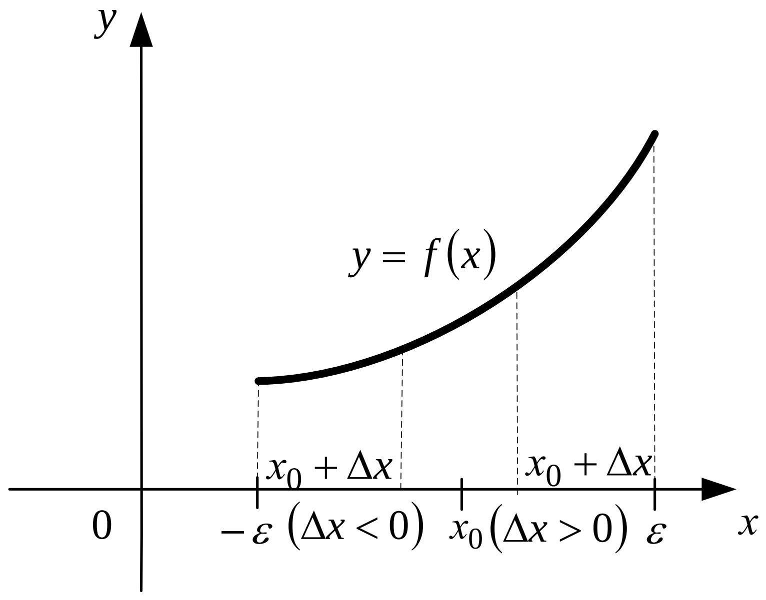

Fig. 12. **The function** $y = f(x)$ **increases at a point** x_0

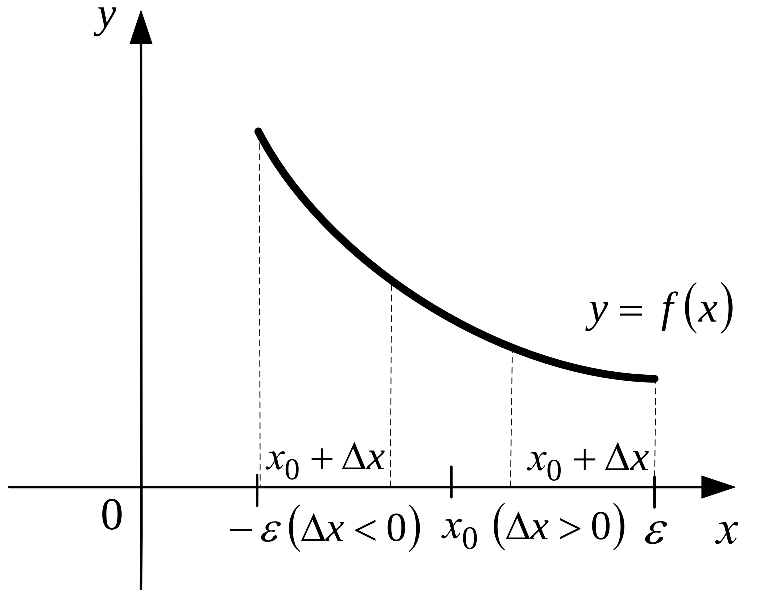

Fig. 13. **The function** $y = f(x)$ **decreases at a point** x_0

Let us consider simple tests for an increase and decrease of a function and also tests for a local extremum.

Conditions for increasing (decreasing) of a function are concluded in the following theorem.

Theorem 1. If function $y = f(x)$ has a positive (negative) derivative at a point x_0, then it increases (decreases) at this point.

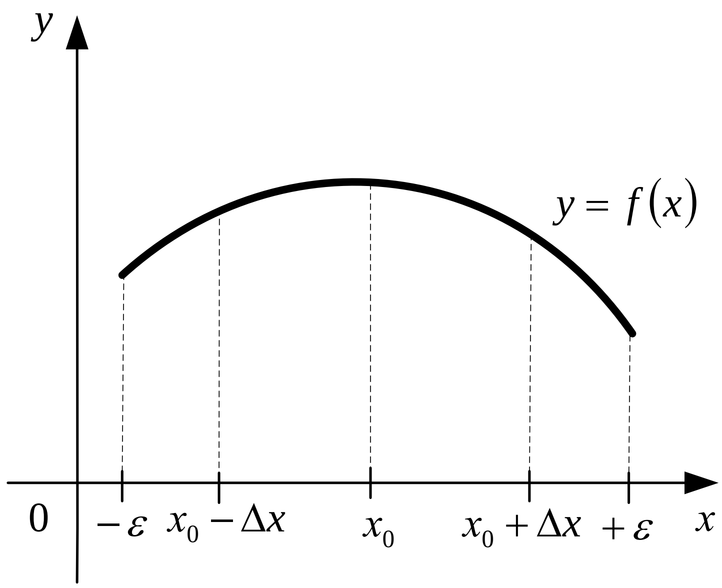

Fig. 14**. The function $y = f(x)$ reaches a local maximum of the point** x_0

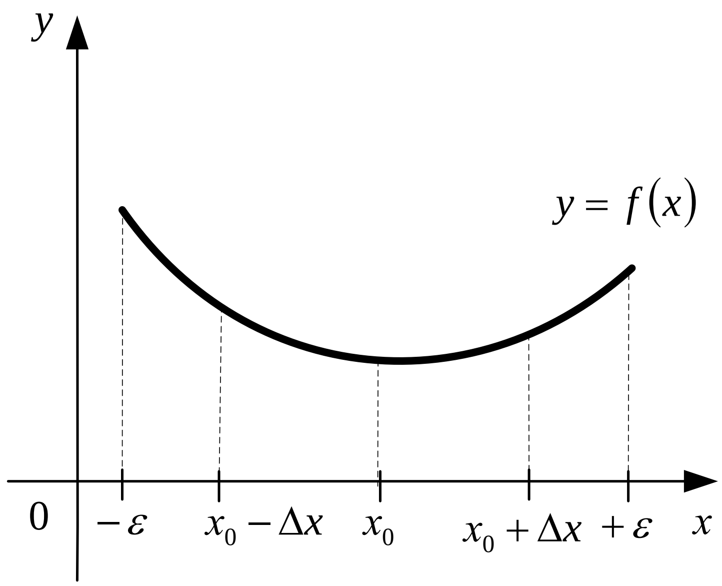

Fig. 15. **The function $y = f(x)$ reaches a local minimum of the point** x_0

Let $f'(x_0) > 0$, then $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0) > 0$ and therefore, the quantity $\frac{\Delta y}{\Delta x}$ has the same sign as $f'(x)$ in some ε-neighborhood of the point x_0. In this ε-neighborhood $\frac{\Delta y}{\Delta x} > 0$, that is, the function $y = f(x)$ increases at the point x_0. The case $f'(x_0) < 0$ is considered in a similar way.

Now we can establish a necessary condition (test) for an extremum.

A necessary Condition for an Extremum. If a function has an extremum at a point x_0, its derivative at that point is either equal to zero or does not exist.

Indeed, if the function attains a maximum at the point x_0 its value at this point is the greatest in the neighborhood of this point x_0. It follows from the previous theorem and from Fermat's theorem. The argument is completely similar in the case of a minimum.

Geometrically, this means that the tangent to the graph of a function is parallel to x-axis at its "tops" and "cavities" (see Fig. 6 Sec. 12).

A function can also have an extremum at some of the points where it is non-differentiable. Examples of that kind have been demonstrated in Fig. 4 and Fig. 5 (Sec. 4).

Definition 4. The points at which the derivative of the function is equal to zero or does not exist are called *critical points*.

It should be noted that the necessary condition for an extremum is not sufficient: the fact that the derivative at a given point turns into zero (or does not exist) does not necessarily imply that this point is a point of an extremum. For example, the derivative of the function $y = x^3$ is $y' = 3x^2$; it vanishes at the point $x = 0$ but this point is not a point of an extremum of the function (Fig. 16).

To find out whether a given point where the derivative turns into zero or does exist is a point of extremum we should resort to sufficient conditions (tests) for an extremum to which we proceed now.

The First Condition for an Extremum in Terms of the First Derivative is based on the following theorem.

Theorem 2. Let the function $f(x)$ be continuous in some neighborhood of a point x_0 and has in this neighborhood a derivative satisfying the following conditions: $f'(x_0) \geq 0$ ($f'(x_0) \leq 0$) for $x > x_0$ and $f'(x_0) \leq 0$ ($f'(x_0) \geq 0$) for $x < x_0$. Then the function $f(x)$ has a local minimum (maximum) at the point x_0.

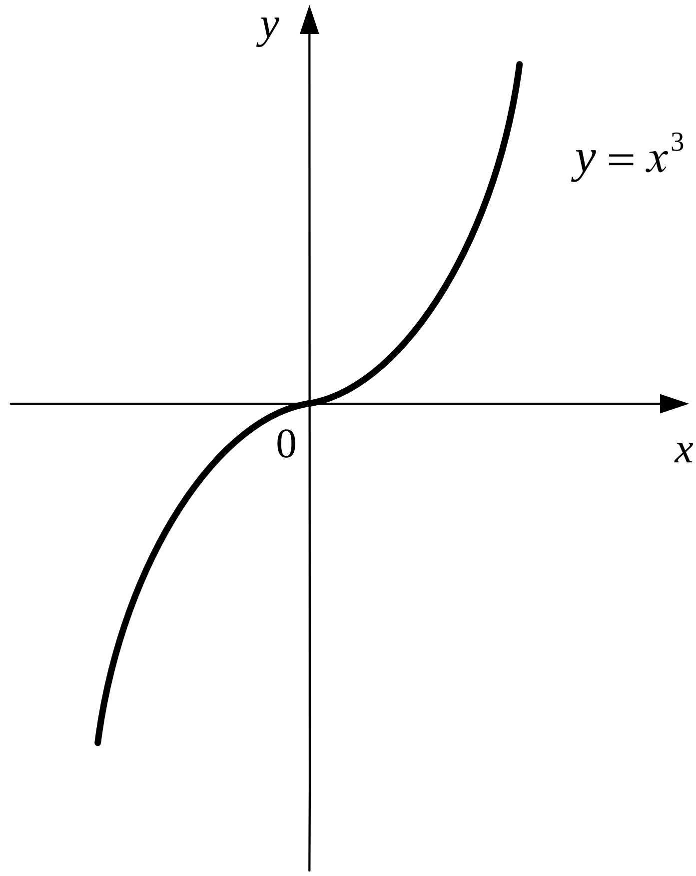

Fig. 16. **The graph of the function** $y = x^3$

By Lagrange′s formula for finite increments, in the considered neighborhood of the point x_0 we have:

$$f(x) - f(x_0) = f'(c)(x - x_0),\ c \in (x; x_0).$$

Therefore, by virtue of the conditions of this theorem both for $x > x_0$ and $x < x_0$ we have: $f'(c)(x - x_0) \geq 0$ ($f'(c)(x - x_0) \leq 0$), that is $f(x) \geq f(x_0)$ ($f(x) \leq f(x_0)$). Consequently, the function $f(x)$ has a local minimum (maximum) at the point x_0.

Thus briefly: if, as x passes through the point x_0 (from left to right), the derivative changes its sign at the point x_0 is a point of extremum.

If the derivative changes the sign from **+** to **–,** there is a maximum at the point x_0, if conversely this is a minimum.

The conditions of this theorem can be simplified if the existence of the second derivative is assumed.

The Second Sufficient Condition for an Extremum in Terms of the Second Derivative is concluded in the theorem.

Theorem 3. Let the function $f(x)$ have a derivative $f'(x)$ in the neighborhood of the point x_0 and a second derivative $f''(x)$ at the point x_0 itself. If $f'(x_0)=0$ and $f''(x_0)>0$ ($f''(x_0)<0$), then the point x_0 is a point of a local minimum (maximum) of the function $f(x)$.

Since the derivative $f'(x)$ exists in the neighborhood of the point x_0, the function $f(x)$ is continuous in this neighborhood.

Let us show that the conditions of Theorem 2 are completely fulfilled. Since $f''(x_0)>0$ ($f''(x_0)<0$), the function $f'(x)$ increases (decreases) at the point x_0. In addition, $f'(x_0)=0$ and, therefore, in some neighborhood of the point x_0, we have: $f'(x)>0$ ($f'(x)<0$) for $x>x_0$ and $f'(x)<0$ ($f'(x)>0$) for $x<x_0$. Thus, it follows from Theorem 2 that for $x=x_0$ the function $f(x)$ has a local minimum (maximum) at the point x_0.

Remark. If $f'(x_0)=0$ and $f''(x_0)=0$, then the problem remains unsolved and it is necessary to resort to the First Condition.

Let us consider several examples.

Example 1. Find intervals of increasing and decreasing of the function $y=f(x)=x-2\sin x$, $x\in[0;2\pi]$.

Solution. Let us find the derivative $y'=1-2\cos x$. It is evident that $y'>0$ in the interval $\left(\frac{\pi}{3};\frac{5\pi}{3}\right)$ and $y'<0$ in the interval $\left(0;\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3};2\pi\right)$. Thus this function increases in the interval $\left(\frac{\pi}{3};\frac{5\pi}{3}\right)$ and decreases in the intervals $\left(0;\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3};\frac{5\pi}{3}\right)$.

Example 2. Find the point of a local extreme of the function $y=f(x)=x^{\alpha}e^{-\beta x}$ and $\alpha>0,\beta>0,x>0$.

Solution. Since $y' = x^{\alpha-1}e^{-\beta x}(\alpha - \beta x)$, we have $y' = 0$ for $x = \frac{\alpha}{\beta}$, $y' > 0$ for $x < \frac{\alpha}{\beta}$, $y' < 0$ for $x > \frac{\alpha}{\beta}$. Therefore, the point $x = \frac{\alpha}{\beta}$ is a point of maximum for the function $y = x^{\alpha}e^{-\beta x}$

Example 3. Find the extremum of the function $y = x\sqrt{1-x^2}$.

Solution. This function is defined as $x \in [-1;1]$. Let us find the derivative $y' = \frac{1-2x^2}{\sqrt{1-x^2}}$.

If $y' = 0$ then $1-2x^2 = 0$, whence $x_1 = -\frac{1}{\sqrt{2}}$, $x_2 = \frac{1}{\sqrt{2}}$ (critical points); $y' = \infty$ as $x = \pm 1$. These points are bound points of the domain of the definition of the function.

Let us find the second derivative $y'' = \frac{x(2x^2-3)}{(1-x^2)^{3/2}}$. Now we compute the values of the second derivative at the critical points. When $x_1 = -\frac{1}{\sqrt{2}}$, we have

$$y''\left(-\frac{1}{\sqrt{2}}\right) = \frac{\left(-\frac{1}{\sqrt{2}}\right)\left(2\left(-\frac{1}{\sqrt{2}}\right)^2 - 3\right)}{\left(1-\left(-\frac{1}{\sqrt{2}}\right)^2\right)^{3/2}} = 4 > 0,$$

hence in accordance with Theorem 3 we conclude that the function has a minimum $y_{\min} = y\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$ at the point $x_1 = -\frac{1}{\sqrt{2}}$.

When $x_2 = \frac{1}{\sqrt{2}}$, we have

$$y''\left(\frac{1}{\sqrt{2}}\right)=\frac{\frac{1}{\sqrt{2}}\left(2\left(\frac{1}{\sqrt{2}}\right)^2-3\right)}{\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)^{3/2}}=-4<0,$$

i.e. the function has a maximum $y_{\max}=y\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2}$ at the point $x_2=\frac{1}{\sqrt{2}}$.

There is no extremum at the critical points $x=\pm 1$ since the points of the extremum can be found only at interior points of the domain of the definition of the function.

Example 4. Prove that $a^2+b^2+c^2\le a^3+b^3+c^3$ if $a>0, b>0, c>0$, $abc=1$.

Solution. Let us

$$f(x)=a^x+b^x+c^x,\ f'(x)=a^x\ln a+b^x\ln b+c^x\ln c,$$

for which we can tell only that

$$f'(0)=a^0\ln a+b^0\ln b+c^0\ln c=\ln a+\ln b+\ln c=\ln abc=\ln 1=0.$$

However, the second derivative is $f''(x)=a^x\ln^2 a+b^x\ln^2 b+c^x\ln^2 c$, which is clearly positive. We thus deduce that $f'(x)$ is increasing, and so $f'(x)\ge f'(0)=0$, for $x\ge 0$, therefore $f(x)$ itself is increasing for $x\ge 0$, and the conclusion follows: $f(2)\le f(3)\Rightarrow a^2+b^2+c^2\le a^3+b^3+c^3$.

Example 5. Determine the most economical dimensions of a closed cylindrical cistern of the given volume V with the least total surface.

Solution. Denoting the radius and the altitude of the cistern in terms of R and H and its total surface as S, we obtain

$$S=2\pi RH+2\pi R^2.$$

Here R and H are not independent variables but they are connected with the relationship

$$V = \pi R^2 H$$

since according to the condition this cistern should have the given volume. We determine the altitude H by this relationship

$$H = \frac{V}{\pi R^2}.$$

Substituting H into the expression of total surface we get

$$S = 2\pi R \frac{V}{\pi R^2} + 2\pi R^2 = 2\left(\pi R^2 + \frac{V}{R}\right), \text{ where } \quad 0 < R < +\infty.$$

Thus writing down $S = S(R)$ we find its least value as $0 < R < \infty$

$$S'(R) = 2\left(2\pi R - \frac{V}{R^2}\right) = 0, \text{ or } \ 2\pi R = \frac{V}{R^2} \Rightarrow R = \sqrt[3]{\frac{V}{2\pi}} \text{ and}$$

$$H = \frac{V}{\pi\left(\frac{V}{2\pi}\right)^{2/3}} = \sqrt[3]{\frac{4V}{\pi}}.$$

This radius R is found in the considered interval

$$S''(R) = 2\left(2\pi + \frac{2V}{R^3}\right) \text{ and } S''\left(\sqrt[3]{\frac{V}{2\pi}}\right) = 12\pi > 0.$$

Whence it follows that $R = \sqrt[3]{\frac{V}{2\pi}}$ is the point of minimum. The function

$S = S(R)$ is continuous $\forall R \in (0; +\infty)$ therefore the single minimum coincides with its least value in this interval. So $R = \sqrt[3]{\dfrac{V}{2\pi}}$ and $H = \sqrt[3]{\dfrac{4V}{\pi}}$ are the most economical dimensions of the cistern.

16. Finding Greatest and Least Values of the Function

Let the function $y = f(x)$ be continuous on a closed interval $[a;b]$. In order to determine the greatest value and the least value of the function the values of the function at the points of extremum should be compared with each other and with the end-point values. For the greatest (the least) value of the function on the interval $[a;b]$ is either one of its maximum (minimum) values or an end-point value.

Example 1. Determine the greatest and the least values of the function

$$y = x^3 - 3x^2 - 9x + 35, \quad x \in [-4; 4].$$

Solution. First of all let us find the critical points or the points at which the first derivative is equal to zero:

$$y' = 3x^2 - 6x - 9 = 0 \Rightarrow x_1 = -1; \ x_2 = 3, \quad x_1 \in [-4; 4] \text{ and } x_2 \in [-4; 4];$$

$$y(x_1) = y(-1) = 40, \ y(x_2) = y(3) = 8.$$

Now we compute the values of the function at the edges of the segment $y(-4) = 41, \ y(4) = 15$.

Comparing all the values of the function at the interior critical points and its values at the edges we can draw the following conclusion.

The greatest value of the function in the interval $[-4; 4]$ is equal to 40 and it attains at the interior point $x = -1$.

The least value is equal to -41 and it reaches at the left-hand edge of the segment $[-4; 4]$ that is the point $x = -4$.

The approximate graph of this function is represented in Fig. 17.

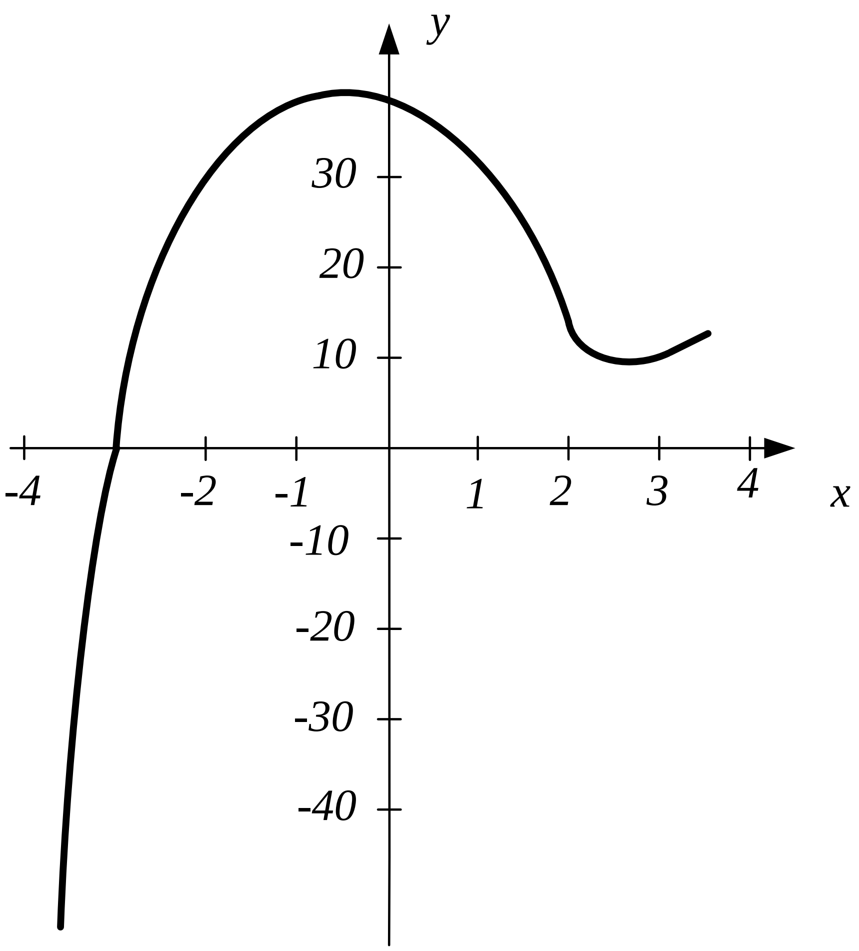

Fig. 17. **The approximate graph of the function** $y(x)$

17. The Direction of Convexity and the Point of Inflection of a Curve

We begin with the following definition.

Definition 1. A curve $y = f(x)$ is said to be convex upward (downward) at a point x_0 if there is a neighborhood of the point x_0 such that for all of the points the curve $y = f(x)$ is situated below (above) the tangent to the curve at the point x_0.

For instance in Fig. 18 the curve $y = f(x)$ is "bulging downward" at the point x_1 and is "bulging upward" at the point x_2.

Definition 2. A point x_0 is said to be a point of inflection of a curve $y = f(x)$ if, when x passes through the value x_0 the moving point of the curve passes from one side of the tangent at the point x_0 to the other side.

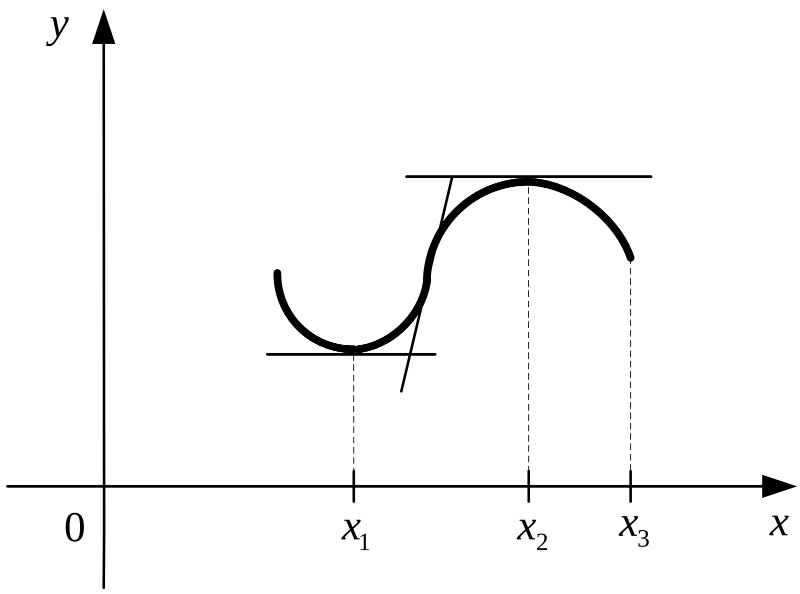

Fig. 18. **The curve $y = f(x)$ is "bulging downward" at the point x_1 and is "bulging upward" at the point x_2**

In other words a point of a curve separating its convex arc from a concave arc is termed a point of inflection. For instance, in Fig. 18, the point x_3 is a point of inflection of the curve $y = f(x)$. At the point of inflection the tangent intersects the curve, in the vicinity of such a point the curve lies on both sides of its tangent drawn through that point.

Let us indicate the basic test defining point of convexity upward (downward) and the point of deflection.

Theorem 1. If the second derivative $f''(x)$ is everywhere negative (positive) within an interval, the arc of the curve $y = f(x)$ corresponding to that interval is convex upward (downward or it is concave).

We present this theorem without proof but we note that it is contained in many textbooks in higher mathematics. From the above argument and Theorem 1 the necessary and sufficient tests (conditions) for a point of inflection follow.

The necessary condition for a point of inflection: at this point $y'' = f''(x_0) = 0$ or does not exist.

It should be noted that by for not every root of the equation $f''(x)=0$ is the abscissa of a point of inflection. For example, the first and the second derivatives of the function $y=x^4$ are $y'=4x^3$ and $y''=12x^2$. Although $y''(x_0)=0 \rightarrow x_0=0$ the point $(0;0)$ is be the vertex of the parabola $y=x^4$ and does not serve as its point of inflection.

In order to find out the point of inflection there is a sufficient test which is contained in the following statement.

If the second derivative $f''(x)$ changes sign as *x* passes through x_0 (from left to right), then x_0 is the abscissa of the point of inflection. If it changes sign from – to +, there is an interval of convexity upward on the left of the point x_0 and an interval of concavity on the right of it (convexity downward), and, conversely, if it changes sign from + to –, an interval of convexity downwards follows an interval of convexity upward as *x* passed through x_0.

Example 1. Investigate the direction of convexity and find the points of inflection $y=arctg\dfrac{1}{x}$.

Solution. Let us find the first derivative

$$y'=\frac{1}{1+\frac{1}{x^2}}\cdot\left(-\frac{1}{x^2}\right)=-\frac{1}{1+x^2}<0.$$

It is a negative value for $\forall x\in(-\infty;+\infty)$. Then the function decreases along the whole numerical axis.

The second derivative

$$y''=\frac{2x}{\left(1+\frac{1}{x^2}\right)^2};\ y''=0\Rightarrow \text{as } x=0.$$

It is evident that for $x<0$ $y''<0$, then the graph of the function is convex upward (sign "∩") and $\forall x\in(0;+\infty)$ the graph of the function is convex downward (sign "U"). Consequently, according to the sufficient test $x_0 = 0$ is the abscissa of the point of inflection. And the interval $\forall x \in (-\infty; 0)$ is the interval of convexity while the interval $\forall x \in (0; +\infty)$ is the interval of concavity.

18. Asymptotes of Curves

Sometimes, the notion of asymptotes is useful when the graph of a function $y = f(x)$ is investigated.

Definition 1. A straight line $y = kx + b$ is called an asymptote to a curve $y = f(x)$ as $x \to +\infty$ ($x \to -\infty$) if

$$\lim_{x \to \pm\infty} \left(f(x) - (kx + b) \right) = 0.$$

This is the definition of the inclined asymptote.

Let us give the necessary and sufficient conditions for an asymptote to exist. Let there exist an asymptote $y = kx + b$ to a curve $y = f(x)$ as $x \to +\infty$. Then

$$\lim_{x \to +\infty} \frac{f(x) - (kx + b)}{x} = 0, \quad \text{whence} \quad k = \lim_{x \to +\infty} \frac{f(x)}{x}.$$

Given k, we can find b from the equality $b = \lim\limits_{x \to \pm\infty} \left(f(x) - kx \right)$. The converse is also true: if the limits k and b exist, then a straight line $y = kx + b$ is an asymptote to the curve $y = f(x)$ as $x \to +\infty$. The condition for an asymptote to exist as $x \to -\infty$ is formulated and proved in a similar manner.

In the case, when $k = 0$ and the equation of an asymptote is $y = b$, then we have a so-called horizontal asymptote. If

$$\lim_{x \to a+0} f(x) = +\infty \text{ (or } -\infty\text{) or } \lim_{x \to a-0} f(x) = +\infty \text{ (or } -\infty\text{)},$$

the curve $y = f(x)$ is sometimes said to have a vertical asymptote for $x = a$.

Example 1. Are there asymptotes for the curve of the function

$$y = f(x) = x \arctan x?$$

Solution. The equation of asymptotes $y = kx + b$, and

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x}, \quad b = \lim_{x \to \pm\infty} \left(f(x) - kx\right).$$

For this function

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x \arctan x}{x} = \lim_{x \to \pm\infty} \arctan x = \pm\frac{\pi}{2}.$$

Then

$$b = \lim_{x \to \pm\infty} \left(f(x) - kx\right) = \lim_{x \to \pm\infty} \left(x \arctan x - \left(\pm\frac{\pi x}{2}\right)\right) = \lim_{x \to \pm\infty} x\left(\arctan x \mp \frac{\pi}{2}\right) = 0.$$

Thus, the curve of the function $f(x) = x \arctan x$ has asymptotes $y = \frac{\pi}{2}x$ and $y = -\frac{\pi}{2}x$.

19. A General Scheme for the Investigation of the Graph of a Function

Above we have considered the methods for a qualitative investigation of functions and their graphs. Now we may recommended the following scheme for investigation of graphs:

1) find the domain of the definition and the range of values of a given function (if possible);

2) find the points at which the graph intercepts the axes of coordinates;

3) determine the character of symmetry of the graph, that is whether the function is even or odd (or neither);

4) determine the periodicity of the graph;

5) find the points of discontinuity, clarify their character and find vertical asymptotes;

6) by the sign of the first derivative, find the intervals of increase and decrease of the function and its points of extremum;

7) by the sign of the second derivative, classify the points of extremum, find the intervals of convexity and concavity and the possible positions of inflection points;

8) determine the behavior of the function at infinity and its inclined (and, in particular, horizontal) asymptotes and draw an approximate graph of the function.

If the computation of some derivatives and the determination of their sings involve many difficulties, then some of the above items may be omitted in the process of investigation.

Example 1. Investigate the graph of the function:

$$y = f(x) = \frac{x^3}{x^2 - 1}.$$

Solution.

1) The domain of the definition of the function:

the intervals $x \in (-\infty;-1) \cup (-1;1) \cup (1;+\infty)$,

the range of values $y \in (-\infty;+\infty)$.

2) $x = 0$; $y = 0$ is the point of intersection with axes of coordinates.

3) Since $f(-x) = -f(x)$, this function is odd. Then it is sufficient to investigate the graph of the function for $x \in (0;+\infty)$.

4) For $x \in [0;1)$ we have $y \le 0$ for $x \in (1;+\infty)$ we have $y > 0$. For $x = 1$ the function has a vertical asymptote (a point of discontinuity of the second kind).

$$\lim_{x \to 1-0} \frac{x^3}{x^2 - 1} = -\infty, \quad \lim_{x \to 1+0} \frac{x^3}{x^2 - 1} = +\infty,$$

$x = -1$ is also the equation of the vertical asymptote since the function is odd.

5) $y' = \dfrac{3x^2(x^2-1) - 2x\,x^3}{(x^2-1)^2} = \dfrac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2} = \dfrac{x^4 - 3x^2}{(x^2-1)^2} = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$.

For $x_1 = 0$, $x_2 = -\sqrt{3}$ and $x_3 = \sqrt{3}$, $y' = 0$. For $x \in (0;\sqrt{3})$ we have $y' < 0$ that is the function decreases, for $x \in (\sqrt{3};+\infty)$, $y' > 0$ the function increases.

Therefore, $x_3 = \sqrt{3}$ is a point of minimum $y_{min}(\sqrt{3}) = \dfrac{3\sqrt{3}}{2} \approx 2{,}55.$

For $x \in \left(-\sqrt{3};0\right)$, $y' < 0$ and we can see that passing through the point $x_1 = 0$ does not change the sign, therefore there is no extremum at the point $x_1 = 0$.

Since the function is odd the point $x_2 = -\sqrt{3}$ is the point of maximum and $y_{max}(-\sqrt{3}) = -\dfrac{3\sqrt{3}}{2} \approx -2{,}55.$

6) $y'' = \left(\dfrac{x^4 - 3x^2}{(x^2-1)^2}\right)' = \dfrac{2x(x^2+3)}{(x^2-1)^3}$.

For $x = 0$, $y'' = 0$ and for $x = -1$ and $x = 1$ does not exist. Therefore, $x_1 = 0$, $x_4 = -1$, $x_5 = 1$ are the points of possible inflection.

Passing through the point $x_1 = 0$ in the interval $x \in \left(-1;1\right)$ y'' changes the sign from "**+**" to "**-**" that is the function is convex downward in the interval $x \in (-1;0]$ and the curve of the function is convex upward for $x \in \left(0;1\right)$. The point $x_1 = 0$ is the point of inflection.

For $x > 1$, $y'' > 0$ and the function is concave that is the point $x_5 = 1$ is also the point of inflection. Analogously $x_4 = -1$ is the point of inflection because the function is odd.

7) Since $k=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\pm\infty}\dfrac{x^3}{x(x^2-1)}=\lim\limits_{x\to\pm\infty}\dfrac{x^2}{x^2-1}=\lim\limits_{x\to\pm\infty}\dfrac{1}{1-\dfrac{1}{x^2}}=1$

and

$$b=\lim_{x\to\pm\infty}\left(f(x)-kx\right)=\lim_{x\to\pm\infty}\left(\frac{x^3}{x^2-1}-1\right)=\lim_{x\to\pm\infty}\frac{x}{x^2-1}=\lim_{x\to\pm\infty}\frac{\frac{1}{x}}{1-\frac{1}{x^2}}=0$$

for $x\to\pm\infty$ the function has the asymptote $y=x$.

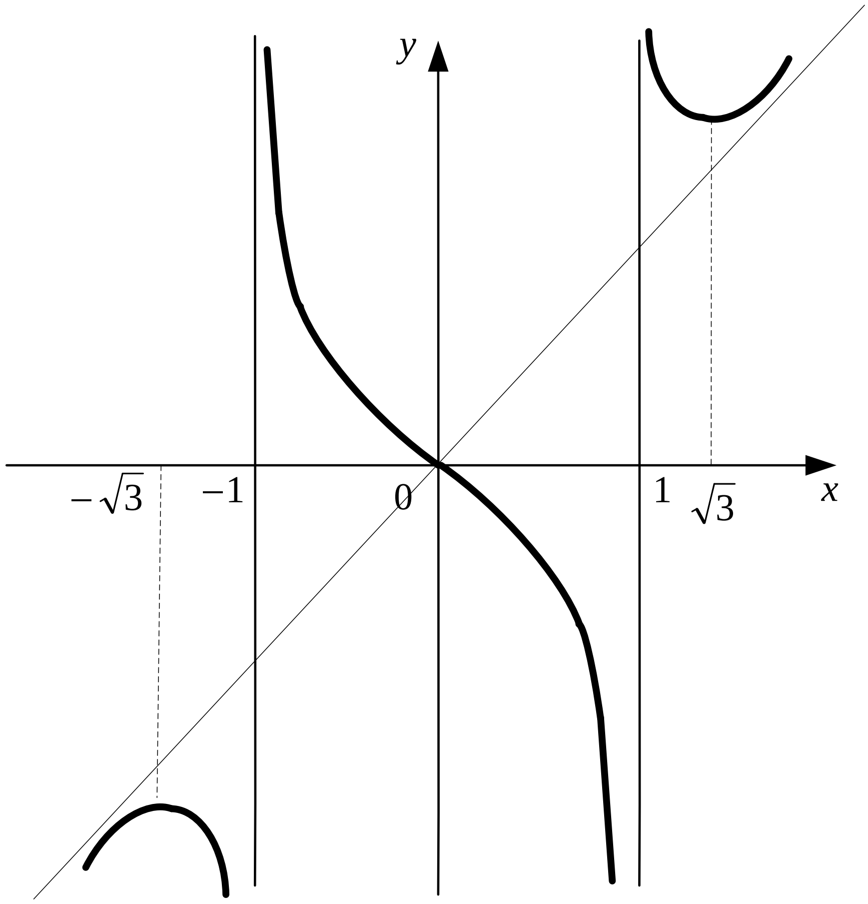

Fig. 19. **The graph of the function** $y=f(x)=\dfrac{x^3}{x^2-1}$

It is convenient to place the results of the investigation into the following table.

Table 1

x	0	$(0;1)$	1	$(1;\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3};+\infty)$
y'	0	–	does not exist	–	0	+
y''	0	–	does not exist	+	+	+
y	0	\ ∩	does not exist	\ U	$y_{\min} = 2{,}55$	/ U

The arrow ↓ in this table indicates the decreasing function and ↑ indicates the increasing function. With the aid of this table we can plot the graph of the function (Fig. 19).

Individual tasks

1 – 4. Find the first and the second derivatives of the given functions.

5. Find the greatest and least values of the function.

6, 7. Investigate the graph of the function.

8. Find an equation of a tangent and a normal to a function at a point.

9, 10. Find limits using L′Hospital′s rule.

Variant 1

1) $y = \arctan\sqrt{x^2 - 1}$;

2) $y = x\sin^2 4x$;

3) $y = \dfrac{\arcsin x}{\ln x}$;

4) $e^y = 4x - 7y$;

5) $y = \ln\left(x^2 - 2x + 2\right)$, $[0; 3]$;

6) $y = \dfrac{x^3 - 2x + 2}{x - 1}$;

7) $y = x\ln x$;

8) $y = \sqrt{x - 4}$, $x_0 = 8$;

9) $\lim\limits_{x\to\infty} \dfrac{x^2}{\ln\left(e^{x^2} + 1\right)}$;

10) $\lim\limits_{x\to 0} (\cos x)^{1/\sin x}$.

Variant 2

1) $y = \ln\cos\dfrac{3}{x}$;

2) $y = \sqrt{1 + \sin^2 x}$;

3) $y = e^{\arctan 2x}\tan 4x$;

4) $\begin{cases} x = \arcsin 2t \\ y = \arccos 2t \end{cases}$;

5) $y = \dfrac{3x}{x^2 + 1}$, $[0; 5]$;

6) $y = \dfrac{x^2}{x^2 - 4}$;

7) $y = \dfrac{3 - x^2}{x + 2}$;

8) $y = x^3 - 2x^2 + 4x$, $x_0 = 2$;

9) $\lim\limits_{x\to 0} x\ln^3 x$;

10) $\lim\limits_{x\to 0} (\cos 4x)^{1/x^2}$.

Variant 3

1) $y = e^{\cos^2 x}$;

2) $y = \dfrac{x^2}{x^2 - 1}$;

3) $y = \arctan \ln\left(x^4 + 1\right)$;

4) $xy = ctg(x - y)$;

5) $y = xe^x$, $[-2; 0]$;

6) $y = \dfrac{x^4}{4} - \dfrac{x^3}{3} - 3x^2 - 5$;

7) $y = x + \dfrac{1}{x}$;

8) $y = 3tg2x + 1$, $x_0 = \dfrac{\pi}{2}$;

9) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin x \cos x}$;

10) $\lim\limits_{x \to 0} \left(\dfrac{1}{x}\right)^{tg\,2x}$.

Variant 4

1) $y = \sqrt{1 + \ln^2 x}$;

2) $y = \dfrac{\arcsin tg2x}{\pi}$;

3) $y = \dfrac{e^{x^3}}{x^2 + 1}$;

4) $\begin{cases} x = 2\cos^3 t \\ y = 2\sin^3 t \end{cases}$;

5) $y = \dfrac{x}{9 - x^2}$, $[-2; 2]$;

6) $y = \dfrac{3x}{\sqrt{x - 1}}$;

7) $y = \dfrac{x^2}{2} + \dfrac{1}{x}$;

8) $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} + x + 4$, $x_0 = 1$;

9) $\lim\limits_{x \to 0} \dfrac{e^{x^2} - \cos x}{\sin x}$;

10) $\lim\limits_{x \to 0} (\sin 2x)^x$.

Variant 5

1) $y = e^{-\sin x} \arctan 2x$;

2) $y = x\sqrt{x^2 + 7}$;

3) $y = \dfrac{1}{\arcsin \ln x}$;

4) $y^3 = tg(x + y)$;

5) $y = \dfrac{\ln x}{x}$, $[1; 4]$;

6) $y = \dfrac{x+1}{(x-1)^2}$;

7) $y = x + e^{-x}$;

8) $y = -3x^2 + 4x + 7$, $x_0 = -1$;

9) $\lim\limits_{x \to 0} \dfrac{\ln \sin 7x}{\ln \sin 2x}$;

10) $\lim\limits_{x \to 0} x^{\sin 2x}$.

Variant 6

1) $y = \dfrac{\sqrt{x^2 + 1}}{x}$;

2) $y = \sin^2 4x \ln x$;

3) $y = e^{\arcsin 4x}$;

4) $\begin{cases} x = \ln\left(t^2 + 4\right) \\ y = arctg\dfrac{t}{2} \end{cases}$;

5) $y = 108x - x^4$, $[-1; 4]$;

6) $y = \dfrac{x^4}{2} - x^2 + 3$;

7) $y = xe^{-\frac{x}{2}}$;

8) $y = \dfrac{x^4}{4} - 7$, $x_0 = -2$;

9) $\lim\limits_{x \to 0} x ctg \pi x$;

10) $\lim\limits_{x \to 1} (1 - x)^{\ln x}$.

Variant 7

1) $y = \frac{\sin^2 3x}{3}$;

2) $y = \ln \cos e^{2x}$;

3) $y = \sqrt[4]{arctg 4x}$;

4) $\sin y = xy^2 + 5$;

5) $y = 2x^3 - 3x^2 - 12x + 1$, $[-2; 2.5]$;

6) $y = \frac{5x^2}{x^2 + 1}$;

7) $y = \frac{x}{\ln x}$;

8) $y = 9x - x^2$, $x_0 = 3$;

9) $\lim\limits_{x \to 0} \arcsin x ctgx$;

10) $\lim\limits_{x \to 0} (1 - \cos x)^x$.

Variant 8

1) $y = x^3 e^{tg 3x}$;

2) $y = \ln \sin \sqrt{x^2 + 1}$;

3) $y = \arcsin \sqrt{1 - x^2}$;

4) $\begin{cases} x = \frac{t}{t^2 + 1}, \\ y = \frac{1}{t^2 + 1} \end{cases}$;

5) $y = x - 2 \ln x$, $[1; e]$;

6) $y = \frac{2x}{1 - x^2}$;

7) $y = \frac{1 + \ln x}{x}$;

8) $y = 2x^3 - 4x^2 - 5x - 3$, $x_0 = 2$;

9) $\lim\limits_{x \to 0} (1 - \cos x) ctgx$;

10) $\lim\limits_{x \to \infty} \left(x + 2^x\right)^{1/x}$.

Variant 9

1) $y=\dfrac{1}{arctg^2 x}$;

2) $y=2^{-x}\cos 4x$;

3) $y=\ln tge^{4x}$;

4) $\sqrt{x}+\sqrt{y}=\sqrt{5}$;

5) $y=\dfrac{1}{4}x^4-\dfrac{2}{3}x^3-\dfrac{3}{2}x^2+2$, $[-2;4]$;

6) $y=\dfrac{x^2-5x}{x-1}$;

7) $y=x+\dfrac{1}{3x^3}$;

8) $y=x^3+2x+1,\quad x_0=-2$;

9) $\lim\limits_{x\to 0}\dfrac{\ln(x-1)}{e^{2x}-e^{-2x}}$;

10) $\lim\limits_{x\to 0}\left(\dfrac{tgx}{x}\right)^{1/x^2}$.

Variant 10

1) $y=\dfrac{\ln^5 x}{x}$;

2) $y=x^2\sin^2 3x$;

3) $y=e^{\arcsin\sqrt{x}}$;

4) $\begin{cases} x=ctg3x \\ y=\dfrac{1}{\sin 3t} \end{cases}$;

5) $y=\dfrac{x^5-8}{x^4}$, $[-3;-1]$;

6) $y=\dfrac{x^2+6}{x^2-1}$;

7) $y=\ln\left(x^2+4\right)$;

8) $y=x^2-5x+8,\quad x_0=3$;

9) $\lim\limits_{x\to 0}\dfrac{e^{5x}-5x-1}{\sin^2 3x}$;

10) $\lim\limits_{x\to 0}(\sin x)^{tgx}$.

Theoretical questions

1. An increment of a function.
2. A derivative of a function.
3. Velocity of rectilinear motion.
4. Right-hand and left-hand derivatives.
5. The notion of a differential.
6. A linear part of an increment of a function.
7. A production function and a marginal cost.
8. A coefficient of elasticity.
9. The Geometrical meaning of a derivative: a tangent and a normal to a function.
10. A differential function at a point.
11. A continuous function.
12. A point of inflection.
13. A derivative of a sum, a product and a quotient.
14. A derivative of a composite function.
15. A derivative of two inverse functions.
16. A parametric representation of a function.
17. Derivatives of elementary functions.
18. A derivative of an implicit function.
19. Logarithmic differentiation of a function.
20. Applying a differential to approximate calculations.
21. A derivative of the second order.
22. Physical interpretation of the second derivative.
23. A differential of the second order.
24. Derivatives and differentials of higher orders.
25. A local minimum and a local maximum.
26. Fermat′s, Rolle′s, Cauchy′s and Lagrange's theorems.
27. Indeterminate forms. L′Hospital′s rule.
28. Increasing and decreasing functions.
29. Necessary and sufficient conditions for an extremum of the function.
30. Finding the greatest and least values of a function.
31. Direction of convexity. Necessary and sufficient conditions for a point of an inflection of a function. The notion of asymptotes.
32. Necessary and sufficient conditions for an asymptote to exist.
33. A general scheme for the investigation of the graph of a function.

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